

Exactly Solvable Quantum Mechanics

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Abstract

A comprehensive review of exactly solvable quantum mechanics is presented with the emphasis of the recently discovered multi-indexed orthogonal polynomials. The main subjects to be discussed are the factorised Hamiltonians, the general structure of the solution spaces of the Schrödinger equation (Crum's theorem and its modifications), the shape invariance, the exact solvability in the Schrödinger picture as well as in the Heisenberg picture, the creation/annihilation operators and the dynamical symmetry algebras, coherent states, various deformation schemes (multiple Darboux transformations) and the infinite families of multi-indexed orthogonal polynomials, the exceptional orthogonal polynomials, and deformed exactly solvable scattering problems.

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1 Introduction

Assuming the rudimentary knowledge of quantum mechanics [1, 2], we start with the *factorised* Hamiltonians (2.5) and the Schrödinger equations (2.2). The general structure of the solution spaces is explored by the intertwining relations and Crum's theorem (2.27)–(2.35) together with its modifications (2.41)–(2.46). The multiple Darboux transformations are discussed generically (2.48)–(2.57). Exact solvability in the Schrödinger picture is explained by

the *shape invariance* (3.1). The *generic eigenvalue formula* (3.2), *unified Rodrigues formulas* (3.3) and the *forward/backward shift operators* (3.13)–(3.14) are deduced. The solvability in the Heisenberg picture is derived based on the *closure relation* (4.4) between the *sinusoidal coordinate* (4.1) and the Hamiltonian. The *creation/annihilation operators* (4.16)–(4.18) are introduced and their connection with the *three term recurrence relations* (4.2) of the orthogonal polynomials is emphasised. The *dynamical symmetry algebras* (4.19)–(4.21) generated by the Hamiltonian and the creation/annihilation operators are also established for all the solvable systems in the Heisenberg picture. New orthogonal polynomials are constructed in §5. The radial oscillator (2.59) and Pöschl-Teller (2.59) systems are rationally extended in terms of the *virtual state wave functions* (5.4)–(5.7). The *multi-indexed orthogonal polynomials* (5.16)–(5.20) are obtained, which are generalisation of the *exceptional orthogonal polynomials* (5.40). The duality (5.47)–(5.49) between the pseudo virtual states (5.9)–(5.11) and the eigenstates is demonstrated. The next topic is exactly solvable scattering problems and their extensions in §6. After the definition of the *scattering amplitudes* (6.1)–(6.3), *reflectionless potentials* (6.14)–(6.19) are derived from the trivial potential $U \equiv 0$ by multiple Darboux transformations. *Multi-indexed scattering amplitudes* (6.25)–(6.27) are expressed in terms of the *asymptotic exponents* (6.21)–(6.22) of the polynomial type seed solutions. As a typical example, various data of the soliton potential (6.28)–(6.35) are presented. The final section is for a summary and comments. Basic symbols, definitions and some formulas are listed in Appendix.

We usually discuss three elementary examples of exactly solvable potentials, the harmonic oscillator, the radial oscillator and the Pöschl-Teller potentials, which are indicated by the initial of the polynomials constituting the corresponding eigenfunctions, that is, the Hermite (H), the Laguerre (L) and the Jacobi (J) polynomials.

Due to the length constraints, we have to concentrate on the systems of single degree of freedom, which are the most basic and best established part of the theory. We apologise to all the authors whose good works could not be referred to in the review due to the lack of space.

2 General Formulation

For the general settings of quantum mechanics, we refer to the standard textbooks [1, 2]. In this review we discuss exactly solvable Schrödinger equations. In other words we present

various methods of constructing such potentials $U(x) \in \mathbf{R}$ for which the eigenvalue problem of the Hamiltonian \mathcal{H}

$$\mathcal{H}\psi(x) = \mathcal{E}\psi(x), \quad \mathcal{H} \stackrel{\text{def}}{=} \sum_j \frac{p_j^2}{2m_j} + U(x),$$

is exactly solvable. We will concentrate on the most fundamental case, that is, the one dimensional quantum mechanics (1-d QM). Generalisation to multi-degrees of freedom cases will be mentioned in appropriate places.

2.1 Problem Setting: 1-d QM

Let us consider one-dimensional QM defined in an interval (x_1, x_2) , in which x_1 and/or x_2 can be infinite. For finite x_j , $j = 1, 2$, the potential must provide an infinite barrier $\lim_{x \rightarrow x_j} U(x) = +\infty$ at that boundary lest the particle *tunnel out* from (x_1, x_2) . This fact provides proper boundary conditions of the wavefunctions. The dynamical variables are the coordinate x and its conjugate momentum p , which is realised as a differential operator $p = -i\hbar \frac{d}{dx} \equiv -i\hbar \partial_x$. Hereafter we adopt the convention $\hbar = 1$ and $2m = 1$ and consider the following Hamiltonian

$$\mathcal{H} \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + U(x), \quad x_1 < x < x_2, \quad (2.1)$$

with a *smooth* potential $U(x) \in \mathbf{C}^\infty$. We also require that the Hamiltonian is *bounded from below*. The *eigenvalue problem* is to find all the discrete eigenvalues $\{\mathcal{E}(n)\}$ and the corresponding eigenfunctions $\{\phi_n(x)\}$

$$\mathcal{H}\phi_n(x) = \mathcal{E}(n)\phi_n(x), \quad n = 0, 1, \dots, \quad (2.2)$$

of the given Hamiltonian \mathcal{H} (2.1). The numbering of the eigenvalues is monotonously increasing:

$$\mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \dots. \quad (2.3)$$

The eigenfunctions are mutually orthogonal

$$(\phi_n, \phi_m) \stackrel{\text{def}}{=} \int_{x_1}^{x_2} \phi_n(x)^* \phi_m(x) dx = h_n \delta_{nm}, \quad 0 < h_n < \infty, \quad n, m = 0, 1, \dots, \quad (2.4)$$

which is a consequence of the *hermiticity* of the Hamiltonian \mathcal{H} , [1, 2]. Then the *oscillation theorem* asserts that the n -th eigenfunction $\phi_n(x)$ has n simple zeros in (x_1, x_2) . In particular

the *ground state eigenfunction* $\phi_0(x)$ has no zero in (x_1, x_2) , and we will choose the convention that it is *positive* $\phi_0(x) > 0$. We also choose all the eigenfunctions to be *real*, $\phi_n(x) \in \mathbf{R}$.

Roughly speaking we encounter two types of problems. The first is *confining potentials* $\lim_{x \rightarrow x_1+0} U(x) = +\infty = \lim_{x \rightarrow x_2-0} U(x)$, which has *infinitely many discrete eigenvalues*. The rest is non-confining and it has *finitely many or infinite¹ discrete eigenvalues* and if $x_1 = -\infty$ and/or $x_2 = +\infty$, *scattering problems* can be considered. The setting of scattering problems will be introduced at the beginning of section 6.

When all the eigenvalues, finite or infinite in number, and the corresponding eigenfunctions can be obtained explicitly, such a potential is called *exactly solvable* [3, 4]. There are some potentials for which only finitely many eigenvalues and eigenfunctions can be obtained explicitly. Such potentials are called *quasi-exactly solvable* [5, 6, 7, 8].

2.2 Factorised Hamiltonian

Let us consider the eigenvalue problem of a given Hamiltonian (2.1) having a finite or semi-infinite number of *discrete energy levels*. The additive constant of the Hamiltonian is so chosen that the ground state energy vanishes, $\mathcal{E}(0) = 0$. That is, the Hamiltonian is *positive semi-definite*. It is a well known theorem in linear algebra that any positive semi-definite hermitian matrix can be factorised as a product of a certain matrix, say \mathcal{A} , and its hermitian conjugate \mathcal{A}^\dagger . As we will see shortly, the Hamiltonians we consider always have factorised forms in one-dimension as well as in higher dimensions.

The Hamiltonian we consider has a simple factorised form [3]

$$\mathcal{H} \stackrel{\text{def}}{=} \mathcal{A}^\dagger \mathcal{A} \quad \text{or} \quad \mathcal{H} \stackrel{\text{def}}{=} \sum_{j=1}^D \mathcal{A}_j^\dagger \mathcal{A}_j \quad \text{in } D \text{ dimensions.} \quad (2.5)$$

The operators \mathcal{A} and \mathcal{A}^\dagger in 1-d QM are:

$$\mathcal{A} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{dw(x)}{dx}, \quad \mathcal{A}^\dagger = -\frac{d}{dx} - \frac{dw(x)}{dx}, \quad w(x) \in \mathbf{R}, \quad \phi_0(x) = e^{w(x)}, \quad (2.6)$$

$$\mathcal{H} = p^2 + U(x), \quad U(x) \stackrel{\text{def}}{=} (\partial_x w(x))^2 + \partial_x^2 w(x), \quad (2.7)$$

in which a real function $w(x)$ is called a *prepotential*. The Hamiltonian of a multi-degrees of freedom system can be constructed in a similar way:

$$\mathcal{A}_j \stackrel{\text{def}}{=} \frac{\partial}{\partial x_j} - \frac{\partial w(x)}{\partial x_j}, \quad \mathcal{A}_j^\dagger = -\frac{\partial}{\partial x_j} - \frac{\partial w(x)}{\partial x_j} \quad (j = 1, \dots, D), \quad \phi_0(x) = e^{w(x)}. \quad (2.8)$$

¹The Coulomb potential is a well-known example of non-confining potential having infinitely many discrete eigenstates.

The prepotential approach is also useful in Calogero-Moser systems [9] in QM [10, 11].

The Schrödinger equation (2.2) is a second order differential equation and the *ground state wavefunction* $\phi_0(x)$ is determined as a zero mode of the operator \mathcal{A} (\mathcal{A}_j) which is a first order equation:

$$\mathcal{A}\phi_0(x) = 0 \quad (\mathcal{A}_j\phi_0(x) = 0, \quad j = 1, \dots, D) \quad \Rightarrow \quad \mathcal{H}\phi_0(x) = 0. \quad (2.9)$$

It should be stressed that the inverse of the zero mode of \mathcal{A} is the zero mode of \mathcal{A}^\dagger :

$$\mathcal{A}^\dagger\phi_0^{-1}(x) = 0 \quad (\mathcal{A}_j^\dagger\phi_0^{-1}(x) = 0, \quad j = 1, \dots, D). \quad (2.10)$$

This fact simply means that a quasi-exactly solvable system with only one known eigenvalue $\mathcal{E}(0) = 0$ and the corresponding eigenfunction $\phi_0(x)$ can always be constructed by an arbitrary positive and smooth square integrable function $\phi_0(x)$.

At the end of this subsection, let us emphasize that any non-vanishing (for example, positive) solution of the original Hamiltonian (2.1)

$$\mathcal{H}\tilde{\phi}(x) = \tilde{\mathcal{E}}\tilde{\phi}(x), \quad \tilde{\mathcal{E}} \in \mathbf{R}, \quad \tilde{\phi}(x) > 0, \quad x \in (x_1, x_2), \quad (2.11)$$

provides a non-singular factorisation of the original Hamiltonian

$$\mathcal{H} = \tilde{\mathcal{A}}^\dagger \tilde{\mathcal{A}} + \tilde{\mathcal{E}}, \quad \tilde{\mathcal{A}} \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{\partial_x \tilde{\phi}(x)}{\tilde{\phi}(x)}, \quad \tilde{\mathcal{A}}^\dagger = -\frac{d}{dx} - \frac{\partial_x \tilde{\phi}(x)}{\tilde{\phi}(x)}. \quad (2.12)$$

2.3 Intertwining Relations: Crum's Theorem

In this subsection we show the general structure of the solution space of 1-d QM. Let us denote by $\mathcal{H}^{[0]}$ the original factorised Hamiltonian \mathcal{H} (2.1), (2.5) and by $\mathcal{H}^{[1]}$ its *partner* (*associated*) Hamiltonian obtained by changing the order of \mathcal{A}^\dagger and \mathcal{A} :

$$\mathcal{H} \equiv \mathcal{H}^{[0]} \stackrel{\text{def}}{=} \mathcal{A}^\dagger \mathcal{A}, \quad \mathcal{H}^{[1]} \stackrel{\text{def}}{=} \mathcal{A} \mathcal{A}^\dagger. \quad (2.13)$$

One simple and most important consequence of the factorised Hamiltonians (2.13) is the *intertwining relations*:

$$\mathcal{A}\mathcal{H}^{[0]} = \mathcal{A}\mathcal{A}^\dagger\mathcal{A} = \mathcal{H}^{[1]}\mathcal{A}, \quad \mathcal{A}^\dagger\mathcal{H}^{[1]} = \mathcal{A}^\dagger\mathcal{A}\mathcal{A}^\dagger = \mathcal{H}^{[0]}\mathcal{A}^\dagger. \quad (2.14)$$

The pair of Hamiltonians $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]}$ are essentially *iso-spectral* and their eigenfunctions $\{\phi_n^{[0]}(x)\}$ and $\{\phi_n^{[1]}(x)\}$ are related by the Darboux-Crum transformations [12, 13]:

$$\mathcal{H}^{[0]}\phi_n^{[0]}(x) = \mathcal{E}(n)\phi_n^{[0]}(x) \quad (n = 0, 1, \dots), \quad \mathcal{A}\phi_0^{[0]}(x) = 0, \quad (2.15)$$

$$\mathcal{H}^{[1]}\phi_n^{[1]}(x) = \mathcal{E}(n)\phi_n^{[1]}(x) \quad (n = 1, 2, \dots), \quad (2.16)$$

$$\phi_n^{[1]}(x) \stackrel{\text{def}}{=} \mathcal{A}\phi_n^{[0]}(x), \quad \phi_n^{[0]}(x) = \frac{\mathcal{A}^\dagger}{\mathcal{E}(n)}\phi_n^{[1]}(x) \quad (n = 1, 2, \dots), \quad (2.17)$$

$$(\phi_n^{[1]}, \phi_m^{[1]}) = \mathcal{E}(n)(\phi_n^{[0]}, \phi_m^{[0]}) \quad (n, m = 1, 2, \dots). \quad (2.18)$$

The associated Hamiltonian $\mathcal{H}^{[1]}$, sometimes called the partner Hamiltonian, has the lowest eigenvalue $\mathcal{E}(1)$ and the state corresponding to $\mathcal{E}(0)$ is missing. In other words, the ground state corresponding to $\mathcal{E}(0)$ is *deleted* by the transformation $\mathcal{H}^{[0]} \rightarrow \mathcal{H}^{[1]}$.

If the ground state energy $\mathcal{E}(1)$ is subtracted from the partner Hamiltonian $\mathcal{H}^{[1]}$, it is again positive semi-definite and can be factorised in terms of new operators $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[1]\dagger}$:

$$\mathcal{H}^{[1]} = \mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]} + \mathcal{E}(1), \quad \mathcal{A}^{[1]}\phi_1^{[1]}(x) = 0. \quad (2.19)$$

By changing the orders of $\mathcal{A}^{[1]\dagger}$ and $\mathcal{A}^{[1]}$, a new Hamiltonian $\mathcal{H}^{[2]}$ is defined:

$$\mathcal{H}^{[2]} \stackrel{\text{def}}{=} \mathcal{A}^{[1]}\mathcal{A}^{[1]\dagger} + \mathcal{E}(1). \quad (2.20)$$

These two Hamiltonians, $\mathcal{H}^{[1]} - \mathcal{E}(1)$ and $\mathcal{H}^{[2]} - \mathcal{E}(1)$, are intertwined by $\mathcal{A}^{[1]}$ and $\mathcal{A}^{[1]\dagger}$:

$$\mathcal{A}^{[1]}(\mathcal{H}^{[1]} - \mathcal{E}(1)) = \mathcal{A}^{[1]}\mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]} = (\mathcal{H}^{[2]} - \mathcal{E}(1))\mathcal{A}^{[1]}, \quad (2.21)$$

$$\mathcal{A}^{[1]\dagger}(\mathcal{H}^{[2]} - \mathcal{E}(1)) = \mathcal{A}^{[1]\dagger}\mathcal{A}^{[1]}\mathcal{A}^{[1]\dagger} = (\mathcal{H}^{[1]} - \mathcal{E}(1))\mathcal{A}^{[1]\dagger}. \quad (2.22)$$

The iso-spectrality of the two Hamiltonians $\mathcal{H}^{[1]}$ and $\mathcal{H}^{[2]}$ and the relationship among their eigenfunctions follow as before:

$$\mathcal{H}^{[2]}\phi_n^{[2]}(x) = \mathcal{E}(n)\phi_n^{[2]}(x) \quad (n = 2, 3, \dots), \quad (2.23)$$

$$\phi_n^{[2]}(x) \stackrel{\text{def}}{=} \mathcal{A}^{[1]}\phi_n^{[1]}(x), \quad \phi_n^{[1]}(x) = \frac{\mathcal{A}^{[1]\dagger}}{\mathcal{E}(n) - \mathcal{E}(1)}\phi_n^{[2]}(x) \quad (n = 2, 3, \dots), \quad (2.24)$$

$$(\phi_n^{[2]}, \phi_m^{[2]}) = (\mathcal{E}(n) - \mathcal{E}(1))(\phi_n^{[1]}, \phi_m^{[1]}) \quad (n, m = 2, 3, \dots), \quad (2.25)$$

$$\mathcal{H}^{[2]} = \mathcal{A}^{[2]\dagger}\mathcal{A}^{[2]} + \mathcal{E}(2), \quad \mathcal{A}^{[2]}\phi_2^{[2]}(x) = 0. \quad (2.26)$$

By the transformation $\mathcal{H}^{[1]} \rightarrow \mathcal{H}^{[2]}$ the state corresponding to $\mathcal{E}(1)$ is deleted.

This process can go on indefinitely by successively deleting the lowest lying energy level:

$$\mathcal{H}^{[s]} \stackrel{\text{def}}{=} \mathcal{A}^{[s-1]}\mathcal{A}^{[s-1]\dagger} + \mathcal{E}(s-1) = \mathcal{A}^{[s]\dagger}\mathcal{A}^{[s]} + \mathcal{E}(s), \quad (2.27)$$

$$\mathcal{H}^{[s]} \phi_n^{[s]}(x) = \mathcal{E}(n) \phi_n^{[s]}(x) \quad (n = s, s+1, \dots), \quad \mathcal{A}^{[s]} \phi_s^{[s]}(x) = 0, \quad (2.28)$$

$$\phi_n^{[s]}(x) \stackrel{\text{def}}{=} \mathcal{A}^{[s-1]} \phi_n^{[s-1]}(x), \quad \phi_n^{[s-1]}(x) = \frac{\mathcal{A}^{[s-1]\dagger}}{\mathcal{E}(n) - \mathcal{E}(s-1)} \phi_n^{[s]}(x) \quad (n = s, s+1, \dots), \quad (2.29)$$

$$(\phi_n^{[s]}, \phi_m^{[s]}) = (\mathcal{E}(n) - \mathcal{E}(s-1)) (\phi_n^{[s-1]}, \phi_m^{[s-1]}) \quad (n, m = s, s+1, \dots). \quad (2.30)$$

The quantities in the s -th step are defined by those in the $(s-1)$ -st step: ($s \geq 1$)

$$w^{[s]}(x) \stackrel{\text{def}}{=} \log |\phi_s^{[s]}(x)|, \quad \phi_n^{[s]}(x) \stackrel{\text{def}}{=} \mathcal{A}^{[s-1]} \phi_n^{[s-1]}(x), \quad (2.31)$$

$$\mathcal{A}^{[s]} \stackrel{\text{def}}{=} \partial_x - \partial_x w^{[s]}(x), \quad \mathcal{A}^{[s]\dagger} = -\partial_x - \partial_x w^{[s]}(x), \quad (2.32)$$

The eigenfunctions at the s -th step have succinct *determinant forms* in terms of the Wronskian: ($n \geq s \geq 0$)

$$\mathbf{W}[f_1, \dots, f_m](x) \stackrel{\text{def}}{=} \det \left(\frac{d^{j-1} f_k(x)}{dx^{j-1}} \right)_{1 \leq j, k \leq m} \quad (\text{Wronskian}), \quad (2.33)$$

$$\mathcal{H}^{[s]} = \mathcal{H}^{[0]} - 2\partial_x^2 \log |\mathbf{W}[\phi_0, \phi_1, \dots, \phi_{s-1}](x)|, \quad (2.34)$$

$$\phi_n^{[s]}(x) = \frac{\mathbf{W}[\phi_0, \phi_1, \dots, \phi_{s-1}, \phi_n](x)}{\mathbf{W}[\phi_0, \phi_1, \dots, \phi_{s-1}](x)}, \quad (2.35)$$

In deriving the determinant formulas (2.34) and (2.35) use is made of the following properties of the Wronskian

$$\mathbf{W}[g f_1, g f_2, \dots, g f_n](x) = g(x)^n \mathbf{W}[f_1, f_2, \dots, f_n](x), \quad (2.36)$$

$$\begin{aligned} & \mathbf{W}[\mathbf{W}[f_1, f_2, \dots, f_n, g], \mathbf{W}[f_1, f_2, \dots, f_n, h]](x) \\ &= \mathbf{W}[f_1, f_2, \dots, f_n](x) \mathbf{W}[f_1, f_2, \dots, f_n, g, h](x) \quad (n \geq 0). \end{aligned} \quad (2.37)$$

Another useful property of the Wronskian is that it is invariant when the derivative $\frac{d}{dx}$ is replaced by an arbitrary ‘covariant derivative’ D_i with an arbitrary smooth function $q_i(x)$:

$$D_i \stackrel{\text{def}}{=} \frac{d}{dx} - q_i(x), \quad \mathbf{W}[f_1, f_2, \dots, f_n](x) = \det(D_{j-1} \cdots D_2 D_1 f_k(x))_{1 \leq j, k \leq n}, \quad (2.38)$$

with $D_{j-1} \cdots D_2 D_1 \big|_{j=1} = 1$. The norm of the s -th step eigenfunctions have a simple uniform expression:

$$(\phi_n^{[s]}, \phi_m^{[s]}) = \prod_{j=0}^{s-1} (\mathcal{E}(n) - \mathcal{E}(j)) \cdot (\phi_n, \phi_m). \quad (2.39)$$

To sum up, we have the following

Theorem. (Crum [13]) *For a given Hamiltonian system $\mathcal{H} \equiv \mathcal{H}^{[0]}$, there are associated Hamiltonian systems $\mathcal{H}^{[1]}, \mathcal{H}^{[2]}, \dots$, as many as the total number of discrete eigenvalues of*

the original system $\mathcal{H}^{[0]}$. They share the same eigenvalues $\{\mathcal{E}(n)\}$ of the original system and the eigenfunctions of $\mathcal{H}^{[j]}$ and $\mathcal{H}^{[j+1]}$ are related linearly by $\mathcal{A}^{[j]}$ and $\mathcal{A}^{[j]\dagger}$.

This situation of the Crum's theorem is illustrated in Fig. 1. If the original system $\mathcal{H}^{[0]}$ is *exactly solvable*, then all the associated systems $\{\mathcal{H}^{[j]}\}$ are also *exactly solvable*.

A quantum mechanical system with a factorised Hamiltonian $\mathcal{H} = \mathcal{A}^\dagger \mathcal{A}$ together with the associated one $\mathcal{H}^{[1]} = \mathcal{A} \mathcal{A}^\dagger$ is sometimes called a ‘supersymmetric’ QM [14, 4]. This seems to be rather a misnomer, since as we have shown the factorised form is generic and it implies no extra symmetry. The iso-spectrality is shared by all the associated Hamiltonians, not merely by the first two. In this connection, the transformation of mapping the s -th to the $(s + 1)$ -st associated Hamiltonian is sometimes called susy transformation. It is also known as the Darboux-Crum transformation. Those covering multi-steps are sometimes called ‘higher derivative’ or ‘nonlinear’ or ‘ \mathcal{N} -fold’ susy transformations [15, 16, 17]. We will discuss generic Darboux transformations in §2.5.

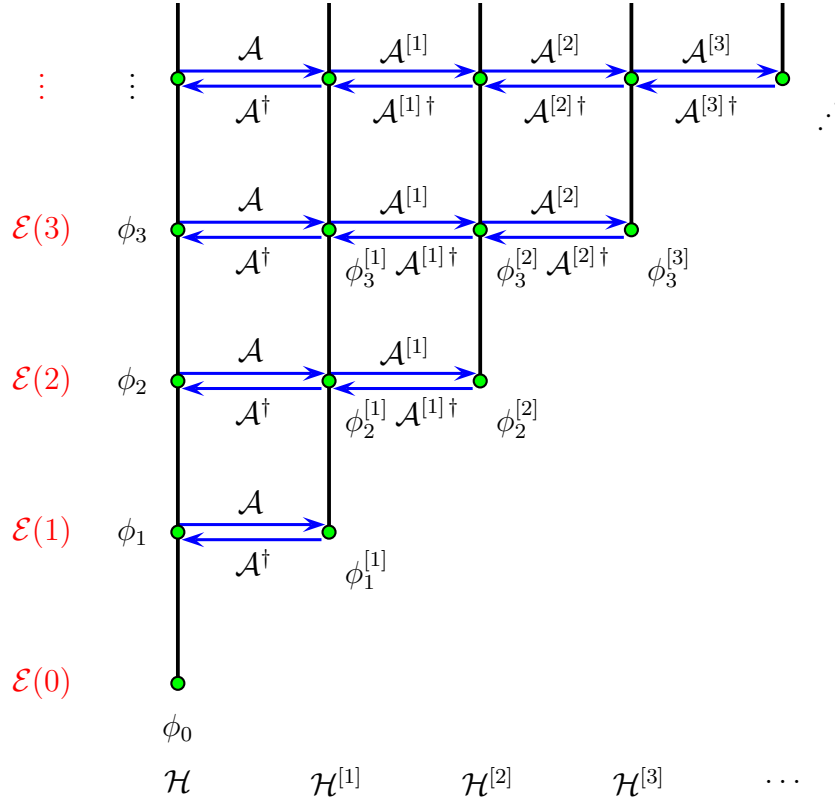


Figure 1: General structure of the solution space of 1-d QM.

2.4 Modified Crum's Theorem

Crum's theorem provides a series of essentially iso-spectral Hamiltonian systems by deleting successively the lowest lying eigenlevels from the original Hamiltonian system \mathcal{H} and $\{\phi_n(x)\}$. The modification of Crum's theorem by Krein-Adler [18, 19] is achieved by deleting a finite number of eigenstates indexed by a set of distinct non-negative integers $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \dots, d_M\} \subset \mathbb{Z}_{\geq 0}^M$, satisfying certain conditions to be specified later (2.47). After the deletion, the new ground state has the label μ given by

$$\mu \stackrel{\text{def}}{=} \min\{n \mid n \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}\}. \quad (2.40)$$

Corresponding to (2.31), the new essentially iso-spectral Hamiltonian system is:

$$\bar{\mathcal{H}}_{\mathcal{D}} \stackrel{\text{def}}{=} \bar{\mathcal{A}}_{\mathcal{D}}^{\dagger} \bar{\mathcal{A}}_{\mathcal{D}} + \mathcal{E}(\mu), \quad \bar{\mathcal{A}}_{\mathcal{D}} \bar{\phi}_{\mathcal{D};\mu}(x) = 0, \quad (2.41)$$

$$\bar{\mathcal{H}}_{\mathcal{D}} \bar{\phi}_{\mathcal{D};n}(x) = \mathcal{E}(n) \bar{\phi}_{\mathcal{D};n}(x) \quad (n \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}), \quad (2.42)$$

$$\bar{\mathcal{H}}_{\mathcal{D}} = p^2 + \bar{U}_{\mathcal{D}}(x), \quad \bar{U}_{\mathcal{D}}(x) \stackrel{\text{def}}{=} U(x) - 2\partial_x^2 \left(\log |W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_M}](x)| \right), \quad (2.43)$$

$$\bar{\mathcal{A}}_{\mathcal{D}} \stackrel{\text{def}}{=} \partial_x - \partial_x \bar{w}_{\mathcal{D}}(x), \quad \bar{\mathcal{A}}_{\mathcal{D}}^{\dagger} = -\partial_x - \partial_x \bar{w}_{\mathcal{D}}(x), \quad \bar{w}_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \log |\bar{\phi}_{\mathcal{D};\mu}(x)|, \quad (2.44)$$

$$\bar{\phi}_{\mathcal{D};n}(x) \stackrel{\text{def}}{=} \frac{W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_M}, \phi_n](x)}{W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_M}](x)}, \quad (2.45)$$

and

$$(\bar{\phi}_n, \bar{\phi}_m) = \prod_{j=1}^{\ell} (\mathcal{E}(n) - \mathcal{E}(d_j)) \cdot (\phi_n, \phi_m) \quad (n, m \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}). \quad (2.46)$$

It should be emphasised that the Hamiltonian $\bar{\mathcal{H}}$ as well as the eigenfunctions $\{\bar{\phi}_n(x)\}$ are symmetric with respect to d_1, \dots, d_M , and thus they are independent of the order of $\{d_j\}$. In order to guarantee the positivity of the norm (2.46) of all the eigenfunctions $\{\bar{\phi}_n(x)\}$ of the modified Hamiltonian, the set of deleted energy levels $\mathcal{D} = \{d_1, \dots, d_M\}$ must satisfy the necessary and sufficient conditions [18, 19]

$$\prod_{j=1}^M (m - d_j) \geq 0, \quad \forall m \in \mathbb{Z}_{\geq 0}. \quad (2.47)$$

And the Hamiltonian $\bar{\mathcal{H}}$ and the potential $\bar{U}(x)$ (2.43) is non-singular under this condition (see [19] for details). Crum's theorem in § 2.3 corresponds to the choice $\{d_1, d_2, \dots, d_M\} = \{0, 1, \dots, M-1\}$. The algebraic derivation of the formulas (2.41)–(2.46) is essentially the same as that of Crum's theorem. See Fig.2 for the illustration of the Krein-Adler transformation.

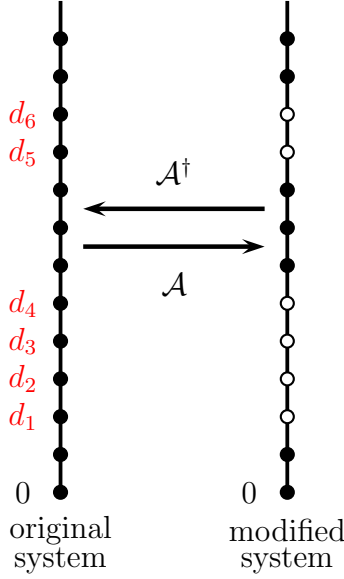


Figure 2: General image of the Krein-Adler transformation. The black circles denote the energy levels, whereas the white circles denote *deleted* energy levels.

Starting from an *exactly solvable* Hamiltonian, one can construct infinitely many variants of *exactly solvable* Hamiltonians and their eigenfunctions by Krein-Adler transformations. The resulting systems are, however, not shape invariant, even if the starting system is.

2.5 Darboux Transformations

Darboux transformations [12] in general apply in much wider contexts than 1-d QM, *i.e.*, without the constraints on the boundary conditions, self-adjointness or factorisation of \mathcal{H} , Hilbert space, etc. In terms of *seed solutions* of a Schrödinger type equation, they map solutions of the same equation to those belonging to a different potential *iso-spectrally*. The potential $U(x)$ in (2.1) can be complex valued and the *seed solutions* $\{\varphi_j(x)\}$ need not be eigenfunctions. Let $\psi(x)$ and $\{\varphi_j(x), \tilde{\mathcal{E}}_j\}$ ($j = 1, \dots, M$) be distinct solutions, not necessarily square-integrable eigenfunctions, of the Schrödinger type equation:

$$\mathcal{H}\psi(x) = \mathcal{E}\psi(x), \quad \mathcal{H}\varphi_j(x) = \tilde{\mathcal{E}}_j\varphi_j(x), \quad \mathcal{E}, \tilde{\mathcal{E}}_j \in \mathbb{C}, \quad j = 1, \dots, M. \quad (2.48)$$

By picking up one seed solution, say, φ_1 , we define new functions

$$\psi^{(1)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \psi](x)}{\varphi_1(x)} \equiv \frac{\varphi_1(x)\psi'(x) - \varphi_1'(x)\psi(x)}{\varphi_1(x)}, \quad (2.49)$$

$$\varphi_1^{(1)}(x) \stackrel{\text{def}}{=} \frac{1}{\varphi_1(x)}, \quad \varphi_k^{(1)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \varphi_k](x)}{\varphi_1(x)}, \quad k = 2, \dots, M. \quad (2.50)$$

It is elementary to show that they are solutions of a new Schrödinger type equation with a deformed Hamiltonian $\mathcal{H}^{(1)}$

$$\mathcal{H}^{(1)} = -\frac{d^2}{dx^2} + U^{(1)}(x), \quad U^{(1)}(x) \stackrel{\text{def}}{=} U(x) - 2\frac{d^2 \log |\varphi_1(x)|}{dx^2}, \quad (2.51)$$

with the same energies \mathcal{E} and $\tilde{\mathcal{E}}_k$:

$$\mathcal{H}^{(1)}\psi^{(1)}(x) = \mathcal{E}\psi^{(1)}(x), \quad \mathcal{H}^{(1)}\varphi_j^{(1)}(x) = \tilde{\mathcal{E}}_j\varphi_j^{(1)}(x), \quad j = 1, \dots, M. \quad (2.52)$$

By repeating these transformations M -times, we arrive at

Theorem. (Darboux [12]) *Let $\psi(x)$ be a solution of the original Schrödinger type equation*

$$\mathcal{H}\psi(x) = \mathcal{E}\psi(x). \quad (2.53)$$

Then the functions

$$\psi^{(M)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \dots, \varphi_M, \psi](x)}{W[\varphi_1, \dots, \varphi_M](x)}, \quad (2.54)$$

$$\varphi_j^{(M)}(x) \stackrel{\text{def}}{=} \frac{W[\varphi_1, \dots, \check{\varphi}_j, \dots, \varphi_M](x)}{W[\varphi_1, \dots, \varphi_M](x)}, \quad j = 1, \dots, M, \quad (2.55)$$

satisfy the M -th deformed Schrödinger type equation with the same energy:

$$\mathcal{H}^{(M)}\psi^{(M)}(x) = \mathcal{E}\psi^{(M)}(x), \quad \mathcal{H}^{(M)}\varphi_j^{(M)}(x) = \tilde{\mathcal{E}}_j\varphi_j^{(M)}(x), \quad j = 1, \dots, M, \quad (2.56)$$

$$\mathcal{H}^{(M)} = -\frac{d^2}{dx^2} + U^{(M)}(x), \quad U^{(M)}(x) \stackrel{\text{def}}{=} U(x) - 2\frac{d^2 \log |W[\varphi_1, \dots, \varphi_M](x)|}{dx^2}. \quad (2.57)$$

Here $\check{\varphi}_j$ in (2.55) means that $\varphi_j(x)$ is excluded from the Wronskian.

In order to apply Darboux transformations in 1-d QM, various restrictions must be imposed on seed solutions so that the deformed potential is non-singular in the physical region. Special choices of seed solutions as eigenfunctions $\varphi_j = \phi_{j-1}$, and $\varphi_j = \phi_{d_j}$, $j = 1, \dots, M$ correspond to Crum [13] and Krein-Adler [18, 19] transformations, respectively.

2.6 Explicit Examples

Here are three *elementary examples of exactly solvable potentials*. The *prepotential* $w(x)$ determines the potential $U(x)$ of the Hamiltonian as shown in (2.7):

$$\text{H: } w(x) = -\frac{1}{2}x^2, \quad \phi_0(x) = e^{-x^2/2}, \quad -\infty < x < \infty,$$

$$\text{harmonic oscillator: } U(x) = x^2 - 1, \quad \eta(x) = x, \quad (2.58)$$

$$\text{L: } w(x) = -\frac{1}{2}x^2 + g \log x, \quad \phi_0(x; g) = e^{-x^2/2}x^g, \quad g > \frac{1}{2}, \quad 0 < x < \infty,$$

$$\text{radial oscillator: } U(x) = x^2 + \frac{g(g-1)}{x^2} - 1 - 2g, \quad \eta(x) = x^2, \quad (2.59)$$

$$\text{J: } w(x) = g \log \sin x + h \log \cos x, \quad \phi_0(x; g, h) = (\sin x)^g (\cos x)^h, \quad g > \frac{1}{2}, \quad h > \frac{1}{2}, \quad 0 < x < \frac{\pi}{2},$$

$$\text{Pöschl-Teller: } U(x) = \frac{g(g-1)}{\sin^2 x} + \frac{h(h-1)}{\cos^2 x} - (g+h)^2, \quad \eta(x) = \cos 2x. \quad (2.60)$$

Their eigenfunctions have a factorised form:

$$\phi_n(x) = \phi_0(x) P_n(\eta(x)), \quad \phi_0(x) = e^{w(x)}, \quad (2.61)$$

in which $P_n(\eta(x))$ is a degree n (except for those discussed in §5) polynomial in the *sinusoidal coordinate* $\eta(x)$. For the above examples, they are the three *classical orthogonal polynomials*; the Hermite (H), the Laguerre (L) and the Jacobi (J) polynomials (for notation, see Appendix):

$$\text{H: } P_n(\eta(x)) = H_n(x) \stackrel{\text{def}}{=} (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{1}{x^2}\right), \quad (2.62)$$

$$\text{L: } P_n(\eta(x)) = L_n^{(g-\frac{1}{2})}(x^2) \stackrel{\text{def}}{=} \frac{(g+\frac{1}{2})_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ g+\frac{1}{2} \end{matrix} \middle| x^2\right), \quad (2.63)$$

$$\text{J: } P_n(\eta(x)) = P_n^{(g-\frac{1}{2}, h-\frac{1}{2})}(\cos 2x) \stackrel{\text{def}}{=} \frac{(g+\frac{1}{2})_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+g+h \\ g+\frac{1}{2} \end{matrix} \middle| \frac{1-\cos 2x}{2}\right). \quad (2.64)$$

The similarity transformed Hamiltonian $\tilde{\mathcal{H}}$ in terms of the ground state wavefunction $\phi_0(x)$ provides the second order equation for $P_n(\eta(x))$

$$\tilde{\mathcal{H}} P_n(\eta(x)) = \mathcal{E}(n) P_n(\eta(x)), \quad \tilde{\mathcal{H}} \stackrel{\text{def}}{=} \phi_0(x)^{-1} \circ \mathcal{H} \circ \phi_0(x) = -\frac{d^2}{dx^2} - 2 \frac{dw(x)}{dx} \frac{d}{dx}. \quad (2.65)$$

The exact solvability can be rephrased as the *lower triangularity* of $\tilde{\mathcal{H}}$

$$\tilde{\mathcal{H}} \eta(x)^n = \mathcal{E}(n) \eta(x)^n + \text{lower orders in } \eta(x), \quad (2.66)$$

in the special basis

$$1, \eta(x), \eta(x)^2, \dots, \eta(x)^n, \dots,$$

spanned by the sinusoidal coordinate $\eta(x)$. This situation is expressed as

$$\tilde{\mathcal{H}} \mathcal{V}_n \subseteq \mathcal{V}_n, \quad \mathcal{V}_n \stackrel{\text{def}}{=} \text{Span}[1, \eta(x), \dots, \eta(x)^n]. \quad (2.67)$$

Obviously, the square of the ground state wavefunction $\phi_0(x)^2 = e^{2w(x)}$ provides the positive definite orthogonality weight function for the polynomials:

$$\int_{x_1}^{x_2} \phi_0(x)^2 P_n(\eta(x)) P_m(\eta(x)) dx = h_n \delta_{nm}, \quad (2.68)$$

$$h_n = \begin{cases} 2^n n! \sqrt{\pi} & : \text{H} \\ \frac{1}{2^n n!} \Gamma(n + g + \frac{1}{2}) & : \text{L} \\ \frac{\Gamma(n + g + \frac{1}{2}) \Gamma(n + h + \frac{1}{2})}{2^n n! (2n + g + h) \Gamma(n + g + h)} & : \text{J} \end{cases}. \quad (2.69)$$

Let us emphasise that the weight function, or $\phi_0(x)$ is determined as a solution of a first order differential equation (2.9).

Let us note that $x = 0$ for L and $x = 0, \pi/2$ for J are the *regular singular points* of second order differential equations. The *monodromy* at the regular singular point is determined by the *characteristic exponent* ρ :

$$M_\rho = e^{2\pi i \rho}. \quad (2.70)$$

The corresponding exponents are expressed simply by the original parameters:

$$\rho = g, 1 - g \text{ for L and } \rho = g, 1 - g (x = 0), \quad \rho = h, 1 - h (x = \pi/2) \text{ for J.} \quad (2.71)$$

It is obvious that the radial oscillator (2.59) and the Pöschl-Teller (2.60) potentials without the constant terms $-(1+2g)$, $-(g+h)^2$ are invariant under the following *discrete symmetry transformations*:

$$\text{L: } g \leftrightarrow 1 - g; \quad \text{J: } g \leftrightarrow 1 - g \text{ and/or } h \leftrightarrow 1 - h. \quad (2.72)$$

The same transformations also keep the above characteristic exponents (2.71) invariant. Likewise, the Hamiltonians of the harmonic (2.58) and the radial (2.59) oscillators (without the constant term) *change sign under the discrete transformation* of the coordinate, $x \rightarrow ix$:

$$\mathcal{H}_h \rightarrow -\mathcal{H}_h, \quad \mathcal{H}_r \rightarrow -\mathcal{H}_r; \quad \mathcal{H}_h \stackrel{\text{def}}{=} -\partial_x^2 + x^2, \quad \mathcal{H}_r \stackrel{\text{def}}{=} -\partial_x^2 + x^2 + \frac{g(g-1)}{x^2}. \quad (2.73)$$

In the next section we will derive these results (2.58)–(2.69) based on *shape invariance*.

3 Shape Invariance: Sufficient Condition of Exact Solvability

Shape invariance [20] is a sufficient condition for the exact solvability in the Schrödinger picture. Combined with Crum's theorem [13], or the factorisation method [3] or the so-

called supersymmetric quantum mechanics [21, 4], the totality of the discrete eigenvalues and the corresponding eigenfunctions can be easily obtained as shown in (3.2) and (3.3).

3.1 Energy Spectrum and Rodrigues Formulas

In many cases the Hamiltonian contains some parameter(s), $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots)$. Here we write the parameter dependence symbolically, $\mathcal{H}(\boldsymbol{\lambda})$, $\mathcal{A}(\boldsymbol{\lambda})$, $\mathcal{E}(n; \boldsymbol{\lambda})$, $\phi_n(x; \boldsymbol{\lambda})$, $P_n(\eta(x); \boldsymbol{\lambda})$ etc, since it is the central issue. The shape invariance condition with a suitable choice of parameters is

$$\mathcal{A}(\boldsymbol{\lambda})\mathcal{A}(\boldsymbol{\lambda})^\dagger = \mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger\mathcal{A}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}(1; \boldsymbol{\lambda}), \quad (3.1)$$

where $\boldsymbol{\delta}$ is the shift of the parameters. In other words $\mathcal{H}^{[0]}$ and $\mathcal{H}^{[1]} - \mathcal{E}(1; \boldsymbol{\lambda})$ have the same shape, only the parameters are shifted by $\boldsymbol{\delta}$. The s -th step Hamiltonian $\mathcal{H}^{[s]}$ in § 2.3 is $\mathcal{H}^{[s]} = \mathcal{H}(\boldsymbol{\lambda} + s\boldsymbol{\delta}) + \mathcal{E}(s; \boldsymbol{\lambda})$. The energy spectrum and the excited state wavefunctions are determined by the data of the ground state wavefunction $\phi_0(x; \boldsymbol{\lambda})$ and the energy of the first excited state $\mathcal{E}(1; \boldsymbol{\lambda})$ as follows [21]:

$$\mathcal{E}(n; \boldsymbol{\lambda}) = \sum_{s=0}^{n-1} \mathcal{E}(1; \boldsymbol{\lambda}^{[s]}), \quad \boldsymbol{\lambda}^{[s]} \stackrel{\text{def}}{=} \boldsymbol{\lambda} + s\boldsymbol{\delta}, \quad (3.2)$$

$$\phi_n(x; \boldsymbol{\lambda}) \propto \mathcal{A}(\boldsymbol{\lambda}^{[0]})^\dagger \mathcal{A}(\boldsymbol{\lambda}^{[1]})^\dagger \mathcal{A}(\boldsymbol{\lambda}^{[2]})^\dagger \dots \mathcal{A}(\boldsymbol{\lambda}^{[n-1]})^\dagger \phi_0(x; \boldsymbol{\lambda}^{[n]}). \quad (3.3)$$

The above formula for the eigenfunctions $\phi_n(x; \boldsymbol{\lambda})$ can be considered as the *universal Rodrigues formula* for the *classical orthogonal polynomials*. For the explicit form of the Rodrigues type formula for each polynomial, one only has to substitute the explicit forms of the operator $\mathcal{A}(\boldsymbol{\lambda})$ and the ground state wavefunction $\phi_0(x; \boldsymbol{\lambda})$.

In the case of a finite number of bound states, e.g. the Morse potential, the eigenvalue has a maximum at a certain level n , $\mathcal{E}(n; \boldsymbol{\lambda})$. Beyond that level the formula (3.2) ceases to work and the Rodrigues formula (3.3) does not provide square integrable eigenfunctions, although ϕ_m ($m > n$) continues to satisfy the Schrödinger equation with $\mathcal{E}(m; \boldsymbol{\lambda})$.

The above shape invariance condition (3.1) is equivalent to the following condition:

$$(\partial_x w(x; \boldsymbol{\lambda}))^2 - \partial_x^2 w(x; \boldsymbol{\lambda}) = (\partial_x w(x; \boldsymbol{\lambda} + \boldsymbol{\delta}))^2 + \partial_x^2 w(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}(1; \boldsymbol{\lambda}). \quad (3.4)$$

It is straightforward to verify the shape invariance for the three examples (2.58)–(2.60) in §2.6 with the following data:

$$\text{H : } \quad \boldsymbol{\lambda} = \phi \text{ (null)}, \quad \boldsymbol{\delta} = \phi, \quad \mathcal{A} = \partial_x + x, \quad \mathcal{E}(n) = 2n, \quad (3.5)$$

$$\text{L: } \boldsymbol{\lambda} = g, \quad \boldsymbol{\delta} = 1, \quad \mathcal{A}(g) = \partial_x + x - g/x, \quad \mathcal{E}(n; g) = 4n, \quad (3.6)$$

$$\text{J: } \boldsymbol{\lambda} = (g, h), \quad \boldsymbol{\delta} = (1, 1), \quad \mathcal{A}(g, h) = \partial_x - g \cot x + h \tan x, \quad (3.7)$$

$$\mathcal{E}(n; g, h) = 4n(n + g + h). \quad (3.8)$$

It should be stressed that the above shape invariant transformation $\boldsymbol{\lambda} \rightarrow \boldsymbol{\lambda} + \boldsymbol{\delta}$, $\mathcal{H}^{[s]} \rightarrow \mathcal{H}^{[s+1]}$ for L and J, that is, $g \rightarrow g + 1$, $h \rightarrow h + 1$, *preserves the monodromy* (2.70) at the regular singularities.

The above universal Rodrigues formula (3.3) for the harmonic oscillator (H) reads

$$e^{-x^2/2} P_n(\eta) \propto (-\partial_x + x)^n e^{-x^2/2}.$$

By using the relation $\partial_x - x = e^{x^2/2} \circ \frac{d}{dx} \circ e^{-x^2/2}$, this gives $P_n(\eta) \propto (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$ and the Rodrigues formula for the Hermite polynomial (A.6) is obtained. The universal Rodrigues formula (3.3) for the radial oscillator (L) reads

$$e^{-x^2/2} x^g P_n(\eta) \propto (-\partial_x + x - \frac{g}{x}) \cdots (-\partial_x + x - \frac{g+n-1}{x}) e^{-x^2/2} x^{g+n}.$$

By using the relation ($\eta = x^2$)

$$-\partial_x + x - \frac{g}{x} = -e^{x^2/2} x^{-g} \circ \frac{d}{dx} \circ e^{-x^2/2} x^g = -2e^{\eta/2} \eta^{-(g-1)/2} \circ \frac{d}{d\eta} \circ e^{-\eta/2} \eta^{g/2},$$

the above formula gives

$$P_n(\eta) \propto (-2)^n e^{\eta} \eta^{-g+1/2} \left(\frac{d}{d\eta}\right)^n (e^{-\eta} \eta^{n+g-1/2}),$$

and the Rodrigues formula for the Laguerre polynomial (A.7) is obtained, up to an n dependent normalisation constant. The verification of the Rodrigues formula (A.8) of the Jacobi polynomial based on (3.3) is left to readers as an exercise.

3.2 Forward and Backward Shift Operators

The shape invariance and the Crum's theorem imply that $\phi_n(x; \boldsymbol{\lambda})$ and $\phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})$ are mapped to each other by the operators $\mathcal{A}(\boldsymbol{\lambda})$ and $\mathcal{A}(\boldsymbol{\lambda})^\dagger$:

$$\mathcal{A}(\boldsymbol{\lambda}) \phi_n(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda}) \phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (3.9)$$

$$\mathcal{A}(\boldsymbol{\lambda})^\dagger \phi_{n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda}) \phi_n(x; \boldsymbol{\lambda}). \quad (3.10)$$

Here the constants $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ depend on the normalisation of $\{\phi_n(x; \boldsymbol{\lambda})\}$ but their product does not. It gives the energy eigenvalue,

$$\mathcal{E}(n; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})b_{n-1}(\boldsymbol{\lambda}). \quad (3.11)$$

For our choice of $\phi_n(x; \boldsymbol{\lambda})$ and $P_n(\eta(x))$ (2.62)–(2.64), the data for $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are:

$$f_n(\boldsymbol{\lambda}) = \begin{cases} 2n & : \text{H} \\ -2 & : \text{L} \\ -2(n+g+h) & : \text{J} \end{cases}, \quad b_{n-1}(\boldsymbol{\lambda}) = \begin{cases} 1 & : \text{H} \\ -2n & : \text{L, J} \end{cases}. \quad (3.12)$$

By removing the ground state contributions, the *forward and backward shift operators* acting on the polynomial eigenfunctions, $\mathcal{F}(\boldsymbol{\lambda})$ and $\mathcal{B}(\boldsymbol{\lambda})$, are introduced:

$$\mathcal{F}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda}) \circ \phi_0(x; \boldsymbol{\lambda}) = c_{\mathcal{F}} \frac{d}{d\eta}, \quad (3.13)$$

$$\mathcal{B}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \phi_0(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}(\boldsymbol{\lambda})^\dagger \circ \phi_0(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \quad (3.14)$$

$$= -4c_{\mathcal{F}}^{-1}c_2(\eta) \left(\frac{d}{d\eta} + \frac{c_1(\eta, \boldsymbol{\lambda})}{c_2(\eta)} \right), \quad (3.15)$$

where $c_{\mathcal{F}}$, $c_1(\eta, \boldsymbol{\lambda})$ and $c_2(\eta)$ are

$$c_{\mathcal{F}} \stackrel{\text{def}}{=} \begin{cases} 1 & : \text{H} \\ 2 & : \text{L} \\ -4 & : \text{J} \end{cases}, \quad c_1(\eta, \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} -\frac{1}{2} & : \text{H} \\ g + \frac{1}{2} - \eta & : \text{L} \\ h - g - (g + h + 1)\eta & : \text{J} \end{cases}, \quad c_2(\eta) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{4} & : \text{H} \\ \eta & : \text{L} \\ 1 - \eta^2 & : \text{J} \end{cases}. \quad (3.16)$$

Then the above relations (3.9)–(3.10) become

$$c_{\mathcal{F}} \frac{d}{d\eta} P_n(\eta; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda}) P_{n-1}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (3.17)$$

$$\mathcal{B}(\boldsymbol{\lambda}) P_{n-1}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda}) P_n(\eta; \boldsymbol{\lambda}). \quad (3.18)$$

Corresponding to (2.5), the forward and backward shift operators give the similarity transformed Hamiltonian (2.65):

$$\tilde{\mathcal{H}}(\boldsymbol{\lambda}) = \mathcal{B}(\boldsymbol{\lambda})\mathcal{F}(\boldsymbol{\lambda}) = -4 \left(c_2(\eta) \frac{d^2}{d\eta^2} + c_1(\eta, \boldsymbol{\lambda}) \frac{d}{d\eta} \right), \quad (3.19)$$

which provides the well known forms of the differential equations for the classical orthogonal polynomials. It is trivial to verify the lower triangularity of $\tilde{\mathcal{H}}$ (2.66) for the three examples (2.62)–(2.64) based on (3.19).

Let us conclude this section by remarks on the effects of Crum §2.3 and Krein-Adler §2.4 transformations of shape invariant systems. By deleting M successive ground states in

terms of Crum transformations, one arrives at the same theory with the shifted parameters, $\lambda \rightarrow \lambda + M\delta$. Thus an essentially new theory is not created by Crum transformations on shape invariant systems. On the other hand, by applying Krein-Adler transformations on a shape invariant system, one can create infinitely new solvable systems as explained in §4.4.

4 Solvability in the Heisenberg Picture

4.1 Closure Relations

As is well known the Heisenberg operator formulation is central to quantum field theory. The creation/annihilation operators of the harmonic oscillators are the cornerstones of modern quantum physics. However, until recently, it had been generally conceived that the Heisenberg operator solutions are intractable. Here we show that most of the shape invariant QM Hamiltonian systems are exactly solvable in the Heisenberg picture, too [22, 23]. To be more precise, the Heisenberg operator of the sinusoidal coordinate $\eta(x)$

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} \quad (4.1)$$

can be evaluated in a closed form. It is well known that any orthogonal polynomials starting at degree 0 satisfy the *three term recurrence relations* [24, 25]

$$\eta P_n(\eta) = A_n P_{n+1}(\eta) + B_n P_n(\eta) + C_n P_{n-1}(\eta) \quad (n \geq 0), \quad (4.2)$$

with $P_0(\eta) = \text{constant}$, $P_{-1}(\eta) = 0$. Here the coefficients A_n , B_n and C_n depend on the normalisation of $\{P_n(\eta)\}$. They are real and $A_{n-1}C_n > 0$ ($n \geq 1$). Conversely all the polynomials starting with degree 0 and satisfying the above three term recurrence relations are orthogonal (Favard's theorem [26]).

For the factorised quantum mechanical eigenfunctions (2.61), these relations mean

$$\eta(x)\phi_n(x) = A_n\phi_{n+1}(x) + B_n\phi_n(x) + C_n\phi_{n-1}(x) \quad (n \geq 0), \quad \phi_{-1}(x) = 0. \quad (4.3)$$

In other words, the operator $\eta(x)$ acts like a creation operator which sends the eigenstate n to $n + 1$ as well as like an annihilation operator, which maps an eigenstate n to $n - 1$. This fact combined with the well known result that the annihilation/creation operators of the harmonic oscillator are the positive/negative frequency parts of the Heisenberg operator solution for the coordinate x is the starting point of this subsection. As will be shown

below the *sinusoidal coordinate* $\eta(x)$ undergoes sinusoidal motion (4.6), whose frequencies depend on the energy. Thus it is not harmonic in general. To the best of our knowledge, the sinusoidal coordinate was first introduced in a rather broad sense for general (not necessarily solvable) potentials as a useful means for coherent state research by Nieto and Simmons [27].

A sufficient condition for the closed form expression of the Heisenberg operator (4.1) is the *closure relation*

$$[\mathcal{H}, [\mathcal{H}, \eta(x)]] = \eta(x) R_0(\mathcal{H}) + [\mathcal{H}, \eta(x)] R_1(\mathcal{H}) + R_{-1}(\mathcal{H}). \quad (4.4)$$

Here the coefficients $R_i(y)$ are polynomials in y . It is easy to see that the cubic commutator $[\mathcal{H}, [\mathcal{H}, [\mathcal{H}, \eta(x)]]] \equiv (\text{ad } \mathcal{H})^3 \eta(x)$ is reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with \mathcal{H} depending coefficients:

$$\begin{aligned} (\text{ad } \mathcal{H})^3 \eta(x) &= [\mathcal{H}, \eta(x)] R_0(\mathcal{H}) + [\mathcal{H}, [\mathcal{H}, \eta(x)]] R_1(\mathcal{H}) \\ &= \eta(x) R_0(\mathcal{H}) R_1(\mathcal{H}) + [\mathcal{H}, \eta(x)] (R_1(\mathcal{H})^2 + R_0(\mathcal{H})) + R_{-1}(\mathcal{H}) R_1(\mathcal{H}), \end{aligned}$$

in which the definition $(\text{ad } \mathcal{H})X \stackrel{\text{def}}{=} [\mathcal{H}, X]$ is used. In this notation the above closure relation (4.4) reads

$$(\text{ad } \mathcal{H})^2 \eta(x) = \eta(x) R_0(\mathcal{H}) + (\text{ad } \mathcal{H}) \eta(x) R_1(\mathcal{H}) + R_{-1}(\mathcal{H}), \quad (4.5)$$

which can be understood as the *Cayley-Hamilton equation* for the operator $\text{ad } \mathcal{H}$ acting on $\eta(x)$. It is trivial to see that all the higher commutators $(\text{ad } \mathcal{H})^n \eta(x)$ can also be reduced to $\eta(x)$ and $[\mathcal{H}, \eta(x)]$ with \mathcal{H} depending coefficients. The second order closure (4.4) simply reflects the Schrödinger equation, which is a second order differential equation. Thus we arrive at

$$\begin{aligned} e^{it\mathcal{H}} \eta(x) e^{-it\mathcal{H}} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} (\text{ad } \mathcal{H})^n \eta(x) \\ &= [\mathcal{H}, \eta(x)] \frac{e^{i\alpha_+(\mathcal{H})t} - e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})} - R_{-1}(\mathcal{H}) R_0(\mathcal{H})^{-1} \\ &\quad + (\eta(x) + R_{-1}(\mathcal{H}) R_0(\mathcal{H})^{-1}) \frac{-\alpha_-(\mathcal{H}) e^{i\alpha_+(\mathcal{H})t} + \alpha_+(\mathcal{H}) e^{i\alpha_-(\mathcal{H})t}}{\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H})}. \end{aligned} \quad (4.6)$$

This simply means that $\eta(x)$ oscillates sinusoidally with two energy-dependent “frequencies” $\alpha_{\pm}(\mathcal{H})$ given by

$$\alpha_{\pm}(\mathcal{H}) \stackrel{\text{def}}{=} \frac{1}{2} (R_1(\mathcal{H}) \pm \sqrt{R_1(\mathcal{H})^2 + 4R_0(\mathcal{H})}), \quad (4.7)$$

$$\alpha_+(\mathcal{H}) + \alpha_-(\mathcal{H}) = R_1(\mathcal{H}), \quad \alpha_+(\mathcal{H})\alpha_-(\mathcal{H}) = -R_0(\mathcal{H}). \quad (4.8)$$

The energy spectrum is determined by the over-determined recursion relations

$$\mathcal{E}(n+1) = \mathcal{E}(n) + \alpha_+(\mathcal{E}(n)), \quad \mathcal{E}(n-1) = \mathcal{E}(n) + \alpha_-(\mathcal{E}(n)), \quad \mathcal{E}(0) = 0. \quad (4.9)$$

It should be stressed that for the known spectra $\{\mathcal{E}(n)\}$ determined by the shape invariance, the quantity inside the square root in the definition of $\alpha_{\pm}(\mathcal{H})$ (4.7) for each n :

$$R_1(\mathcal{E}(n))^2 + 4R_0(\mathcal{E}(n))$$

becomes a complete square and the the above two conditions are consistent. For 1-d QM, the Hamiltonians and the sinusoidal coordinates satisfying the closure relation (4.4) are classified and then the eigenfunctions have the factorised form (2.61) [22]. For the three elementary examples (2.58)–(2.60) given in §2.6, the data are:

$$\text{H : } R_1(y) = 0, \quad R_0(y) = 4, \quad R_{-1}(y) = 0, \quad (4.10)$$

$$A_n = 1/2, \quad B_n = 0, \quad C_n = 1, \quad (4.11)$$

$$\text{L : } R_1(y) = 0, \quad R_0(y) = 16, \quad R_{-1}(y) = -8(y + 2g + 1), \quad (4.12)$$

$$A_n = -(n+1), \quad B_n = (2n + g + 1/2), \quad C_n = -(n + g - 1/2), \quad (4.13)$$

$$\text{J : } R_1(y) = 8, \quad R_0(y) = 16(y + (g+h)^2 - 1), \quad R_{-1}(y) = 16(g-h)(g+h-1), \quad (4.14)$$

$$A_n = \frac{2(n+1)(n+g+h)}{(2n+g+h)(2n+g+h+1)}, \quad B_n = \frac{(h-g)(g+h-1)}{(2n+g+h-1)(2n+g+h+1)},$$

$$C_n = \frac{2(n+g-1/2)(n+h-1/2)}{(2n+g+h-1)(2n+g+h)}. \quad (4.15)$$

It is straight forward to verify the recursion relations (4.9) for the eigenvalue formulas $\mathcal{E}(n)$, (3.5)–(3.8) for the three elementary examples.

For 1-d QM, the necessary and sufficient conditions for the existence of the sinusoidal coordinate satisfying the closure relation (4.4) are analysed in Appendix A of [22]. It was shown that such systems constitute a sub-group of the shape invariant 1-d QM. We also mention that exact Heisenberg operator solutions for independent sinusoidal coordinates as many as the degree of freedom were derived for the Calogero systems based on any root system [28]. These are novel examples of infinitely many multi-particle Heisenberg operator solutions.

4.2 Annihilation and Creation Operators

The *annihilation* and *creation* operators $a^{(\pm)}$ are extracted from this exact Heisenberg operator solution:

$$e^{it\mathcal{H}}\eta(x)e^{-it\mathcal{H}} = a^{(+)}e^{i\alpha_+(\mathcal{H})t} + a^{(-)}e^{i\alpha_-(\mathcal{H})t} - R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}, \quad (4.16)$$

$$\begin{aligned} a^{(\pm)} &\stackrel{\text{def}}{=} \pm \left([\mathcal{H}, \eta(x)] - (\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1})\alpha_{\mp}(\mathcal{H}) \right) (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H}))^{-1} \\ &= \pm (\alpha_+(\mathcal{H}) - \alpha_-(\mathcal{H}))^{-1} \left([\mathcal{H}, \eta(x)] + \alpha_{\pm}(\mathcal{H})(\eta(x) + R_{-1}(\mathcal{H})R_0(\mathcal{H})^{-1}) \right), \end{aligned} \quad (4.17)$$

$$a^{(+)\dagger} = a^{(-)}, \quad a^{(+)}\phi_n(x) = A_n\phi_{n+1}(x), \quad a^{(-)}\phi_n(x) = C_n\phi_{n-1}(x). \quad (4.18)$$

It should be stressed that the annihilation and the creation operators are hermitian conjugate of each other and they act on the eigenstate (4.18). The excited state wavefunctions $\{\phi_n(x)\}$ are obtained by the successive action of the creation operator $a^{(+)}$ on the ground state wavefunction $\phi_0(x)$. This is the exact solvability in the Heisenberg picture.

4.3 Dynamical Symmetry Algebras and Coherent States

Simple commutation relations

$$[\mathcal{H}, a^{(\pm)}] = a^{(\pm)}\alpha_{\pm}(\mathcal{H}), \quad (4.19)$$

follow from (4.17) and (4.4). Commutation relations of $a^{(\pm)}$ are expressed in terms of the coefficients of the three term recurrence relations by (4.18):

$$\begin{aligned} a^{(-)}a^{(+)}\phi_n &= A_nC_{n+1}\phi_n, \quad a^{(+)}a^{(-)}\phi_n = C_nA_{n-1}\phi_n, \\ \Rightarrow [a^{(-)}, a^{(+)}]\phi_n &= (A_nC_{n+1} - A_{n-1}C_n)\phi_n. \end{aligned} \quad (4.20)$$

These relations simply mean the operator relations

$$a^{(-)}a^{(+)} = f(\mathcal{H}), \quad a^{(+)}a^{(-)} = g(\mathcal{H}), \quad (4.21)$$

in which f and g are analytic functions of \mathcal{H} explicitly given for each example. In other words, \mathcal{H} and $a^{(\pm)}$ form a *dynamical symmetry algebra*, which is also called a *quasi-linear algebra* [29]. It should be stressed that the situation is quite different from those of the wide variety of proposed annihilation/creation operators for various quantum systems [30], most of which were introduced within the framework of ‘algebraic theory of coherent states,’ without exact solvability. In all such cases there is no guarantee for symmetry relations like

(4.21). The explicit form of the annihilation operator (4.17) allows us to define the *coherent state* as its eigenvector, $a^{(-)}\psi(\alpha, x) = \alpha\psi(\alpha, x)$, $\alpha \in \mathbb{C}$. See [22] for various examples of coherent states.

4.4 Bochner's Theorem

In 1884 [31], Routh showed that polynomials satisfying the three term recurrence relations and a second order differential equation were one of the *classical* polynomials, the Hermite, Laguerre, Jacobi and Bessel. Later in 1929 [32], Bochner classified all polynomial solutions to second order Sturm-Liouville operators with polynomial coefficients and arrived at the same conclusions. See §20.1 of [25] for more details. This was a kind of No-Go theorem in 1-d QM, since it declared that no essentially new exactly solvable 1-d QM could be achieved as the solutions of the ordinary Schrödinger equation under the assumption of the factorised eigenfunctions (2.61) with n specifying the degree of the polynomial. Avoiding Bochner's theorem was one of the strongest motivations for the introduction of the Askey scheme of hypergeometric orthogonal polynomials and their q -analogues [24, 25, 33]. The *discrete quantum mechanics* [34] was created as a quantum mechanical reformulation of these Askey scheme of classical orthogonal polynomials. In the discrete quantum mechanics Schrödinger equations are difference equations instead of differential, and the constraints by Bochner's theorem do not apply. Many exactly solvable examples of discrete quantum mechanics have been constructed [35, 36].

One simple way to avoid Bochner's theorem in 1-d QM is to apply Krein-Adler transformations on the three examples of classical orthogonal polynomials H, L and J, (2.62)–(2.64) introduced in §2.6. For $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \dots, d_M\} \subset \mathbb{Z}_{\geq 0}^M$, $\mathcal{D} \neq \{0, 1, \dots, M-1\}$, the resulting eigenfunction (2.45) is [19]:

$$\bar{\phi}_{\mathcal{D};n}(x) \stackrel{\text{def}}{=} \frac{W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_M}, \phi_n](x)}{W[\phi_{d_1}, \phi_{d_2}, \dots, \phi_{d_M}](x)} = \psi_{\mathcal{D}}(x) \frac{\mathcal{P}_{\mathcal{D};n}(\eta)}{\Xi_{\mathcal{D}}(\eta)}, \quad \eta \equiv \eta(x), \quad (4.22)$$

$$\psi_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \phi_0(x) (\eta'(x))^{M+1}, \quad \eta'(x) \equiv d\eta(x)/dx, \quad (4.23)$$

$$\Xi_{\mathcal{D}}(\eta) \stackrel{\text{def}}{=} W[P_{d_1}, P_{d_2}, \dots, P_{d_M}](\eta), \quad \mathcal{P}_{\mathcal{D};n}(\eta) \stackrel{\text{def}}{=} W[P_{d_1}, P_{d_2}, \dots, P_{d_M}, P_n](\eta), \quad (4.24)$$

in which a formula

$$W[f_1(\eta(x)), f_2(\eta(x)), \dots, f_n(\eta(x))](x) = (\eta'(x))^{n(n+1)/2} W[f_1, f_2, \dots, f_n](\eta) \quad (4.25)$$

is used. Here $\Xi_{\mathcal{D}}(\eta)$ and $\mathcal{P}_{\mathcal{D};n}(\eta)$ are polynomials in η of degree $\ell_{\mathcal{D}}$ and $\ell_{\mathcal{D}} + n - M$, with $\ell_{\mathcal{D}} \stackrel{\text{def}}{=} \sum_{j=1}^M d_j - M(M-1)/2 \geq M$. Thus the eigenfunctions of the deformed system provide orthogonal polynomials over (x_1, x_2) , $\{\mathcal{P}_{\mathcal{D};n}(\eta)\}$, $n \in \mathbb{Z}_{\geq 0} \setminus \mathcal{D}$, satisfying second order differential equations. The weight function is $W_{\mathcal{D}}(x) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}^2(x)/\Xi_{\mathcal{D}}^2(\eta)$. These polynomials, however, have M ‘holes’ in the degree at $n = d_j$, $j = 1, \dots, M$ and the lowest degree is $\ell_{\mathcal{D}} + \mu - M \geq 0$, in which μ is defined in (2.40).

The simplest example of $\mathcal{D} = \{1, 2\}$ for the harmonic oscillator (H) potential, $W_{\mathcal{D}}(x) \propto e^{-x^2}/(1+2x^2)^2$ was derived by Dubov, Eleonskii and Klagin [37] based on a pseudo virtual state $\tilde{\phi}_2(x)$ (5.9) deletion a few years before Adler [19]. This is the simplest example of the *duality between the pseudo virtual states and eigenstates* demonstrated in §5.3.

In this context, it is clear that orthogonal polynomials avoiding Bochner’s constraints should start at degree $\ell \geq 1$, if they do not have ‘holes’ in the degrees. Thus they do not satisfy the three term recurrence relations (4.2). The pursuit for such new orthogonal polynomials led Gómez-Ullate, Kamran and Milson [38] in 2008 to the discovery of *exceptional orthogonal polynomials*, called X_1 Laguerre and Jacobi polynomials, which start at degree 1. Almost immediately, Quesne proposed quantum mechanical reformulation of the X_1 Laguerre and Jacobi polynomials as the main parts of the eigenfunctions of shape invariant systems [39]. Odake and Sasaki constructed X_{ℓ} Laguerre and Jacobi polynomials, which start at degree ℓ , for all positive integers ℓ [40]. Then Quesne [41] discovered the second type of exceptional Laguerre and Jacobi polynomials at $\ell = 2$. At $\ell = 1$ the first and the second type are identical. These two types are called $X_{\ell}^{\text{I(II)}}$ -L(J) polynomials for short and they are generated by the discrete symmetry transformations (2.72)–(2.73) of the radial oscillator and the Pöschl-Teller potentials. Odake and Sasaki [42] constructed the X_{ℓ}^{II} -Laguerre and Jacobi polynomials for all positive integers ℓ . Soon this exciting hot topic attracted many authors and the rich structures of the subjects [43]–[80], including for example, identities satisfied by orthogonal polynomials [46, 72], Fuchsian differential equations aspects [47, 62, 66, 69, 59], Darboux transformations [48, 49, 55, 57], alternatives of the three term recurrence relations [49, 73], etc were revealed.

5 New Orthogonal Polynomials

In this section we present infinitely many new orthogonal polynomials satisfying second order differential equations, not following the historical developments, but adhering to the logical structure. The main focus is the *multi-indexed* Laguerre and Jacobi polynomials, which include the exceptional orthogonal polynomials as the simplest one-indexed cases. The multiple Darboux transformations with polynomial type seed solutions, which are obtained from the eigenfunctions through discrete symmetry transformations, *the virtual state wave functions*, play the central role. Needless to say, one can apply the Krein-Adler transformations to multi-indexed orthogonal polynomials to generate the ones with the ‘holes’ in the degree.

5.1 Polynomial Type Seed Solutions

For rational extensions of solvable potentials, we need *polynomial type seed solutions*, which are factorised into a prefactor times a polynomial as (2.61). We will introduce three kinds of polynomial type seed solutions, called *virtual state wavefunctions*, *pseudo virtual state wavefunctions* and *overshoot eigenfunctions*. We will discuss overshoot eigenfunctions in connection with the deformations of exactly solvable scattering problems in §6.

The seed solutions $\{\varphi_j(x), \tilde{\mathcal{E}}_j\}$ ($j = 1, 2, \dots, M$) satisfying the following conditions are called *virtual state wavefunctions*:

1. No zeros in $x_1 < x < x_2$, i.e. $\varphi_j(x) > 0$ or $\varphi_j(x) < 0$ in $x_1 < x < x_2$.
2. Negative energy, $\tilde{\mathcal{E}}_j < 0$.
3. $\varphi_j(x)$ is also a polynomial type solution.
4. Square non-integrability, $(\varphi_j, \varphi_j) = \infty$.
5. Reciprocal square non-integrability, $(\varphi_j^{-1}, \varphi_j^{-1}) = \infty$.

These conditions are not totally independent. The negative energy condition is necessary for the positivity of the norm as seen from the norm formula (2.46), since a similar formula is valid for the virtual state wavefunction cases when \mathcal{E}_{d_j} is replaced by $\tilde{\mathcal{E}}_j$.

When the first condition is dropped and the reciprocal is required to be locally square integrable *at both boundaries*, see (5.3), such seed solutions are called *pseudo virtual state wavefunctions*. When the system is extended in terms of a pseudo virtual state wavefunction

$\varphi_j(x)$, the new Hamiltonian $\mathcal{H}^{(1)}$ has an extra *eigenstate* $\varphi_j^{-1}(x)$ with the eigenvalue $\tilde{\mathcal{E}}_j$, *if the new potential is non-singular*. The extra state is below the original ground state and $\mathcal{H}^{(1)}$ is no longer iso-spectral with \mathcal{H} . This is a consequence of (2.52). Its non-singularity is not guaranteed, either. When extended in terms of M pseudo virtual state wavefunctions $\{\varphi_j(x), \tilde{\mathcal{E}}_j\}$ ($j = 1, 2, \dots, M$), the resulting Hamiltonian $\mathcal{H}^{(M)}$ has M additional *eigenstates* $\varphi_j^{(M)}(x)$ (2.55), *if the potential $U^{(M)}(x)$ is non-singular*. They are all below the original ground state.

Since $\varphi_j(x)$ is finite in $x_1 < x < x_2$, the non-square integrability can only be caused by the boundaries. Thus the virtual state wavefunctions belong to either of the following type I and II:

$$\begin{aligned} \text{Type I virtual : } & \int_{x_1}^{x_1+\epsilon} dx \varphi_j(x)^2 < \infty, & \int_{x_2-\epsilon}^{x_2} dx \varphi_j(x)^2 = \infty, \\ & \& \int_{x_1}^{x_1+\epsilon} dx \varphi_j(x)^{-2} = \infty, & \int_{x_2-\epsilon}^{x_2} dx \varphi_j(x)^{-2} < \infty, \end{aligned} \quad (5.1)$$

$$\begin{aligned} \text{Type II virtual : } & \int_{x_1}^{x_1+\epsilon} dx \varphi_j(x)^2 = \infty, & \int_{x_2-\epsilon}^{x_2} dx \varphi_j(x)^2 < \infty, \\ & \& \int_{x_1}^{x_1+\epsilon} dx \varphi_j(x)^{-2} < \infty, & \int_{x_2-\epsilon}^{x_2} dx \varphi_j(x)^{-2} = \infty, \end{aligned} \quad (5.2)$$

$$\begin{aligned} \text{pseudo virtual : } & \int_{x_1}^{x_1+\epsilon} dx \varphi_j(x)^2 = \infty \text{ or } \int_{x_2-\epsilon}^{x_2} dx \varphi_j(x)^2 = \infty, \\ & \& \int_{x_1}^{x_1+\epsilon} dx \varphi_j(x)^{-2} < \infty, & \int_{x_2-\epsilon}^{x_2} dx \varphi_j(x)^{-2} < \infty. \end{aligned} \quad (5.3)$$

An appropriate modification is needed when $x_2 = +\infty$ and/or $x_1 = -\infty$.

5.1.1 Virtual state wavefunctions for L and J

Here we present the explicit forms of the virtual state wave functions for the radial oscillator (L) (2.59) and the Pöschl-Teller (J) potentials (2.60). They are obtained from their eigenfunctions (2.61), (2.63)–(2.64) by the discrete symmetry transformations (2.72) and (2.73) for L1: The virtual states wavefunctions for L are:

$$\begin{aligned} \text{L1 : } & \tilde{\phi}_v^{\text{I}}(x) \stackrel{\text{def}}{=} e^{\frac{1}{2}x^2} x^g \xi_v^{\text{I}}(\eta(x); g), & \xi_v^{\text{I}}(\eta; g) \stackrel{\text{def}}{=} P_v(-\eta; g), \\ & \tilde{\mathcal{E}}_v^{\text{I}} \stackrel{\text{def}}{=} -4(g + v + \frac{1}{2}), & v \in \mathbb{Z}_{\geq 0}, & \tilde{\delta}^{\text{I}} \stackrel{\text{def}}{=} -1, \end{aligned} \quad (5.4)$$

$$\begin{aligned} \text{L2 : } & \tilde{\phi}_v^{\text{II}}(x) \stackrel{\text{def}}{=} e^{-\frac{1}{2}x^2} x^{1-g} \xi_v^{\text{II}}(\eta(x); g), & \xi_v^{\text{II}}(\eta; g) \stackrel{\text{def}}{=} P_v(\eta; 1 - g), \\ & \tilde{\mathcal{E}}_v^{\text{II}} \stackrel{\text{def}}{=} -4(g - v - \frac{1}{2}), & v = 0, 1, \dots, [g - \frac{1}{2}]', & \tilde{\delta}^{\text{II}} \stackrel{\text{def}}{=} 1, \end{aligned} \quad (5.5)$$

in which $[a]'$ denotes the greatest integer less than a and $\tilde{\delta}^{I,II}$ will be used later. For no-nodeness of $\xi_v^{I,II}$, see (2.39) of [46]. The virtual state wavefunctions for J are:

$$\begin{aligned} \text{J1 : } \quad \tilde{\phi}_v^I(x) &\stackrel{\text{def}}{=} (\sin x)^g (\cos x)^{1-h} \xi_v^I(\eta(x); g, h), \quad \xi_v^I(\eta; g, h) \stackrel{\text{def}}{=} P_v(\eta; g, 1-h), \\ \tilde{\mathcal{E}}_v^I &\stackrel{\text{def}}{=} -4(g+v+\frac{1}{2})(h-v-\frac{1}{2}), \quad v=0, 1, \dots, [h-\frac{1}{2}]', \quad \tilde{\delta}^I \stackrel{\text{def}}{=} (-1, 1), \end{aligned} \quad (5.6)$$

$$\begin{aligned} \text{J2 : } \quad \tilde{\phi}_v^{II}(x) &\stackrel{\text{def}}{=} (\sin x)^{1-g} (\cos x)^h \xi_v^{II}(\eta(x); g, h), \quad \xi_v^{II}(\eta; g, h) \stackrel{\text{def}}{=} P_v(\eta; 1-g, h), \\ \tilde{\mathcal{E}}_v^{II} &\stackrel{\text{def}}{=} -4(g-v-\frac{1}{2})(h+v+\frac{1}{2}), \quad v=0, 1, \dots, [g-\frac{1}{2}]', \quad \tilde{\delta}^{II} \stackrel{\text{def}}{=} (1, -1). \end{aligned} \quad (5.7)$$

The larger the parameter g and/or h become, the more the virtual states are ‘created’. The L1 system is obtained from the J1 in the limit $h \rightarrow \infty$. This explains the infinitely many virtual states of L1. Let us denote by $\mathcal{V}^{I,II}$ the index sets of the virtual states of type I and II for L and J. Due to the parity property of the Jacobi polynomial $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$, the two virtual state polynomials ξ_v^I and ξ_v^{II} for J are related by $\xi_v^{II}(-\eta; g, h) = (-1)^v \xi_v^I(\eta; h, g)$. For no-nodeness of $\xi_v^{I,II}$, see (2.40) of [46] and (3.2) of [50]. The label 0 is special in that the wavefunctions satisfy $\tilde{\phi}_0^I(x; \boldsymbol{\lambda}) \tilde{\phi}_0^{II}(x; \boldsymbol{\lambda}) = c_{\mathcal{F}}^{-1} \frac{d\eta(x)}{dx}$ since $\xi_0^{I,II} = 1$. Here the constant $c_{\mathcal{F}} = 2$ for L and $c_{\mathcal{F}} = -4$ for J. We will not use the label 0 states for deletion.

Next we show that the virtual state solutions at the first step $\{\tilde{\phi}_v^{(1)}(x)\}$ have no node in the interior. By using the Schrödinger equations for them we obtain

$$\partial_x W[\tilde{\phi}_d, \tilde{\phi}_v](x) = (\tilde{\mathcal{E}}_d - \tilde{\mathcal{E}}_v) \tilde{\phi}_d(x) \tilde{\phi}_v(x), \quad (5.8)$$

which has no node. Since we can verify that $W[\tilde{\phi}_d, \tilde{\phi}_v](x)$ vanishes at one boundary, no-nodeness of $W[\tilde{\phi}_d, \tilde{\phi}_v](x)$ in the interior follows. When the virtual states d and v belong to different types (I and II) the Wronskians might not vanish at both boundaries. In that case we can show that they have the same sign at both boundaries $W[\tilde{\phi}_d, \tilde{\phi}_v](x_1)W[\tilde{\phi}_d, \tilde{\phi}_v](x_2) > 0$. Likewise we can show, for all the explicit examples in the next section that $\tilde{\phi}_v^{(1)}$ and $1/\tilde{\phi}_v^{(1)}$ have infinite norms. The steps going from $\mathcal{H}^{(1)} \rightarrow \mathcal{H}^{(2)}$ and further are essentially the same. A schematic picture illustrating the virtual state deletion is shown in Fig.3.

5.1.2 Pseudo virtual state wavefunctions for H, L and J

For completeness we list here the pseudo virtual state wave functions for the harmonic oscillator (H), the radial oscillator (L) and the Pöschl-Teller (J) potentials:

$$\text{H : } \quad \tilde{\phi}_v \stackrel{\text{def}}{=} e^{\frac{1}{2}x^2} \xi_v(x), \quad \xi_v(x) \stackrel{\text{def}}{=} i^{-v} H_v(ix)$$

$$\tilde{\mathcal{E}}_v \stackrel{\text{def}}{=} -2(v+1) = \mathcal{E}(-(v+1)), \quad v \in \mathbb{Z}_{\geq 0}, \quad \tilde{\delta} \stackrel{\text{def}}{=} -1, \quad (5.9)$$

$$\begin{aligned} \text{L : } \quad \tilde{\phi}_v &\stackrel{\text{def}}{=} e^{\frac{1}{2}x^2} x^{1-g} \xi_v(\eta(x); g), \quad \xi_v(\eta; g) \stackrel{\text{def}}{=} P_v(-\eta; 1-g), \\ \tilde{\mathcal{E}}_v &\stackrel{\text{def}}{=} -4(v+1) = \mathcal{E}(-(v+1)), \quad v \in \mathbb{Z}_{\geq 0}, \quad \tilde{\delta} \stackrel{\text{def}}{=} -1, \end{aligned} \quad (5.10)$$

$$\begin{aligned} \text{J : } \quad \tilde{\phi}_v(x) &\stackrel{\text{def}}{=} (\sin x)^{1-g} (\cos x)^{1-h} \xi_v(\eta(x); g, h), \quad \xi_v(\eta; g, h) \stackrel{\text{def}}{=} P_v(\eta; 1-g, 1-h), \\ \tilde{\mathcal{E}}_v &\stackrel{\text{def}}{=} -4(v+1)(g+h-v-1) = \mathcal{E}(-(v+1)), \quad v \in \mathbb{Z}_{\geq 0}, \quad \tilde{\delta} \stackrel{\text{def}}{=} (-1, -1). \end{aligned} \quad (5.11)$$

5.2 Multi-indexed Orthogonal Polynomials

Here we recapitulate the multi-indexed Laguerre and Jacobi polynomials as presented in [57]. The basic formula is (2.54) in Theorem in §2.5.

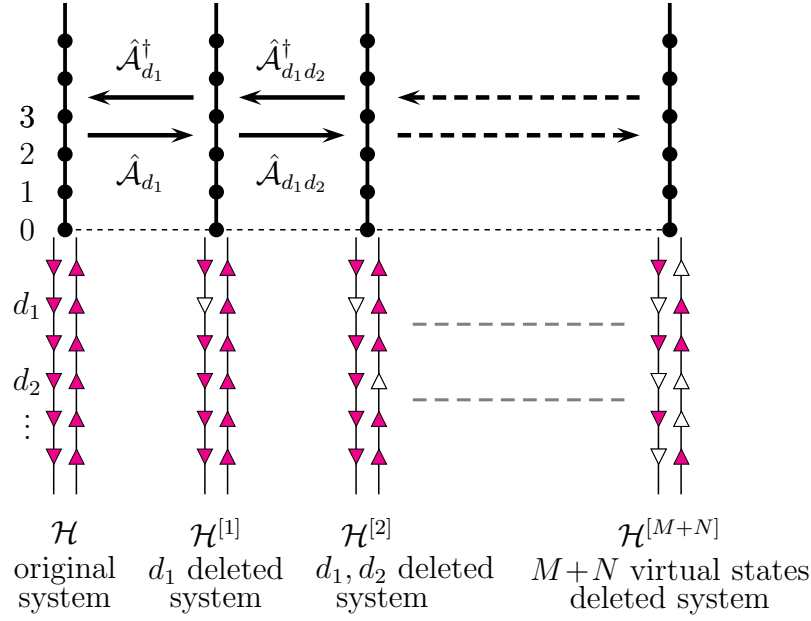


Figure 3: Schematic picture of virtual states deletion. The black circles denote eigenstates. The down and up triangles denote virtual states of type I and II. The deleted virtual states are denoted by white triangles.

Since there are two types of virtual states available, the general deletion is specified by the set of $M+N$ positive integers $\mathcal{D} \stackrel{\text{def}}{=} \{d_1^{\text{I}}, \dots, d_M^{\text{I}}, d_1^{\text{II}}, \dots, d_N^{\text{II}}\}$, $d_j^{\text{I,II}} \geq 1$. In order to accommodate all these virtual states, the parameters g and h must be larger than certain bounds:

$$\text{L : } \quad g > \max\{N + \frac{3}{2}, d_j^{\text{II}} + \frac{1}{2}\}, \quad (5.12)$$

$$\text{J : } \quad g > \max\{N + 2, d_j^{\text{II}} + \frac{1}{2}\}, \quad h > \max\{M + 2, d_j^{\text{I}} + \frac{1}{2}\}. \quad (5.13)$$

Like the original eigenfunctions (2.61) the n -th eigenfunction $\phi_{\mathcal{D},n}(x; \boldsymbol{\lambda}) \equiv \phi_n^{[M,N]}(x)$ after the deletion (2.54) can be clearly factorised into an x -dependent part, the common denominator polynomial $\Xi_{\mathcal{D}}$ in η and the *multi-indexed polynomial* $P_{\mathcal{D},n}$ in η :

$$\phi_n^{[M,N]}(x) \equiv \phi_{\mathcal{D},n}(x; \boldsymbol{\lambda}) = c_{\mathcal{F}}^{M+N} \psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) P_{\mathcal{D},n}(\eta(x); \boldsymbol{\lambda}), \quad \psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{\phi_0(x; \boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta(x); \boldsymbol{\lambda})}, \quad (5.14)$$

in which $\phi_0(x; \boldsymbol{\lambda})$ is the ground state wavefunction (2.59)-(2.60). Here the shifted parameters $\boldsymbol{\lambda}^{[M,N]}$ after the $[M, N]$ deletion is $\boldsymbol{\lambda}^{[M,N]} \stackrel{\text{def}}{=} \boldsymbol{\lambda} - M\tilde{\boldsymbol{\delta}}^{\text{I}} - N\tilde{\boldsymbol{\delta}}^{\text{II}}$, explicitly it is

$$\boldsymbol{\lambda}^{[M,N]} = g + M - N \quad \text{for L}, \quad \boldsymbol{\lambda}^{[M,N]} = (g + M - N, h - M + N) \quad \text{for J}. \quad (5.15)$$

The polynomials $P_{\mathcal{D},n}$ and $\Xi_{\mathcal{D}}$ are expressed in terms of Wronskians in the variable η :

$$P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} W[\mu_1, \dots, \mu_M, \nu_1, \dots, \nu_N, P_n](\eta) \times \begin{cases} e^{-M\eta} \eta^{(M+g+\frac{1}{2})N} & : \text{L} \\ \left(\frac{1-\eta}{2}\right)^{(M+g+\frac{1}{2})N} \left(\frac{1+\eta}{2}\right)^{(N+h+\frac{1}{2})M} & : \text{J} \end{cases}, \quad (5.16)$$

$$\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} W[\mu_1, \dots, \mu_M, \nu_1, \dots, \nu_N](\eta) \times \begin{cases} e^{-M\eta} \eta^{(M+g-\frac{1}{2})N} & : \text{L} \\ \left(\frac{1-\eta}{2}\right)^{(M+g-\frac{1}{2})N} \left(\frac{1+\eta}{2}\right)^{(N+h-\frac{1}{2})M} & : \text{J} \end{cases}, \quad (5.17)$$

$$\mu_j = \begin{cases} e^{\eta} \xi_{d_j^{\text{I}}}^{\text{I}}(\eta; g) & : \text{L} \\ \left(\frac{1+\eta}{2}\right)^{\frac{1}{2}-h} \xi_{d_j^{\text{I}}}^{\text{I}}(\eta; g, h) & : \text{J} \end{cases}, \quad \nu_j = \begin{cases} \eta^{\frac{1}{2}-g} \xi_{d_j^{\text{II}}}^{\text{II}}(\eta; g) & : \text{L} \\ \left(\frac{1-\eta}{2}\right)^{\frac{1}{2}-g} \xi_{d_j^{\text{II}}}^{\text{II}}(\eta; g, h) & : \text{J} \end{cases}, \quad (5.18)$$

in which P_n in (5.16) denotes the original polynomial, $P_n(\eta; g)$ for L and $P_n(\eta; g, h)$ for J. The multi-indexed polynomial $P_{\mathcal{D},n}$ is of degree $\ell + n$ and the denominator polynomial $\Xi_{\mathcal{D}}$ is of degree ℓ in η , in which ℓ is given by

$$\ell \stackrel{\text{def}}{=} \sum_{j=1}^M d_j^{\text{I}} + \sum_{j=1}^N d_j^{\text{II}} - \frac{1}{2}M(M-1) - \frac{1}{2}N(N-1) + MN \geq 1. \quad (5.19)$$

Here the label n specifies the energy eigenvalue $\mathcal{E}(n; \boldsymbol{\lambda})$ of $\phi_{\mathcal{D},n}$ and it also counts the nodes due to the oscillation theorem in §2.1. The multi-indexed polynomials $\{P_{\mathcal{D},n}\}$ form a complete set of orthogonal polynomials with the orthogonality relations:

$$\int d\eta \frac{W(\eta; \boldsymbol{\lambda}^{[M,N]})}{\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda})^2} P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda}) P_{\mathcal{D},m}(\eta; \boldsymbol{\lambda}) = h_n(\boldsymbol{\lambda}) \delta_{nm} \times \begin{cases} \prod_{j=1}^M (n + g + d_j^{\text{I}} + \frac{1}{2}) \cdot \prod_{j=1}^N (n + g - d_j^{\text{II}} - \frac{1}{2}) & : \text{L} \\ 4^{-M-N} \prod_{j=1}^M (n + g + d_j^{\text{I}} + \frac{1}{2})(n + h - d_j^{\text{I}} - \frac{1}{2}) \\ \quad \times \prod_{j=1}^N (n + g - d_j^{\text{II}} - \frac{1}{2})(n + h + d_j^{\text{II}} + \frac{1}{2}) & : \text{J} \end{cases}, \quad (5.20)$$

where the weight function of the original polynomials $W(\eta; \boldsymbol{\lambda})d\eta = \phi_0(x; \boldsymbol{\lambda})^2 dx$ reads explicitly

$$W(\eta; \boldsymbol{\lambda}) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2} e^{-\eta} \eta^{g-\frac{1}{2}} & : \text{L} \\ \frac{1}{2^{g+h+1}} (1-\eta)^{g-\frac{1}{2}} (1+\eta)^{h-\frac{1}{2}} & : \text{J} \end{cases}. \quad (5.21)$$

5.2.1 Deformed Hamiltonians

We explore various properties of the new multi-indexed polynomials $\{P_{\mathcal{D},n}\}$. The lowest degree polynomial $P_{\mathcal{D},0}(\eta; \boldsymbol{\lambda})$ is related to the denominator polynomial $\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda})$ by the parameter shift $\boldsymbol{\lambda} \rightarrow \boldsymbol{\lambda} + \boldsymbol{\delta}$ ($\boldsymbol{\delta} = 1$ for L and $\boldsymbol{\delta} = (1, 1)$ for J):

$$P_{\mathcal{D},0}(\eta; \boldsymbol{\lambda}) = \Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta}) \times \begin{cases} (-1)^M \prod_{j=1}^N (g - d_j^{\text{II}} - \frac{1}{2}) & : \text{L} \\ 2^{-M} \prod_{j=1}^M (h - d_j^{\text{I}} - \frac{1}{2}) \cdot (-2)^{-N} \prod_{j=1}^N (g - d_j^{\text{II}} - \frac{1}{2}) & : \text{J} \end{cases}. \quad (5.22)$$

The Hamiltonian $\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \equiv \mathcal{H}^{[M,N]}$ of the $[M, N]$ deleted system can be expressed in terms of its ground state eigenfunction $\phi_{\mathcal{D},0}(x; \boldsymbol{\lambda}) \equiv \phi_0^{[M,N]}(x; \boldsymbol{\lambda})$ with the help of (5.22):

$$\mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) = \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}), \quad (5.23)$$

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \frac{d}{dx} - \frac{\partial_x \phi_{\mathcal{D},0}(x; \boldsymbol{\lambda})}{\phi_{\mathcal{D},0}(x; \boldsymbol{\lambda})}, \quad \phi_{\mathcal{D},0}(x; \boldsymbol{\lambda}) \propto \phi_0(x; \boldsymbol{\lambda}^{[M,N]}) \frac{\Xi_{\mathcal{D}}(\eta(x); \boldsymbol{\lambda} + \boldsymbol{\delta})}{\Xi_{\mathcal{D}}(\eta(x); \boldsymbol{\lambda})}. \quad (5.24)$$

Reflecting the construction [49, 50] it is shape invariant [20, 40, 46]

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger = \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda} + \boldsymbol{\delta})^\dagger \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda} + \boldsymbol{\delta}) + \mathcal{E}(1; \boldsymbol{\lambda}). \quad (5.25)$$

This means that the operators $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})$ and $\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger$ relate the eigenfunctions of neighbouring degrees and parameters:

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \phi_{\mathcal{D},n}(x; \boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda}) \phi_{\mathcal{D},n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}), \quad (5.26)$$

$$\mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger \phi_{\mathcal{D},n-1}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda}) \phi_{\mathcal{D},n}(x; \boldsymbol{\lambda}), \quad (5.27)$$

in which the constants $f_n(\boldsymbol{\lambda})$ and $b_{n-1}(\boldsymbol{\lambda})$ are the factors of the eigenvalue $f_n(\boldsymbol{\lambda}) b_{n-1}(\boldsymbol{\lambda}) = \mathcal{E}(n; \boldsymbol{\lambda})$ given in (3.12). The forward and backward shift operators are defined by

$$\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x; \boldsymbol{\lambda}) \quad (5.28)$$

$$= c_{\mathcal{F}} \frac{\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda})} \left(\frac{d}{d\eta} - \frac{\partial_\eta \Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})}{\Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda} + \boldsymbol{\delta})} \right), \quad (5.29)$$

$$\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda}) \stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x; \boldsymbol{\lambda})^{-1} \circ \mathcal{A}_{\mathcal{D}}(\boldsymbol{\lambda})^\dagger \circ \psi_{\mathcal{D}}(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \quad (5.30)$$

$$= -4c_{\mathcal{F}}^{-1}c_2(\eta)\frac{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta})}\left(\frac{d}{d\eta}+\frac{c_1(\eta,\boldsymbol{\lambda}^{[M,N]})}{c_2(\eta)}-\frac{\partial_{\eta}\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}\right), \quad (5.31)$$

in which $c_1(\eta;\boldsymbol{\lambda})$ and $c_2(\eta)$ are given in (3.16). Their action on the multi-indexed polynomials $P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda})$ is

$$\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) = f_n(\boldsymbol{\lambda})P_{\mathcal{D},n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}), \quad (5.32)$$

$$\mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})P_{\mathcal{D},n-1}(\eta;\boldsymbol{\lambda}+\boldsymbol{\delta}) = b_{n-1}(\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}). \quad (5.33)$$

5.2.2 Second Order Equations for the New Polynomials

The second order differential operator $\tilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})$ governing the multi-indexed polynomials is:

$$\begin{aligned} \tilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda}) &\stackrel{\text{def}}{=} \psi_{\mathcal{D}}(x;\boldsymbol{\lambda})^{-1} \circ \mathcal{H}_{\mathcal{D}}(\boldsymbol{\lambda}) \circ \psi_{\mathcal{D}}(x;\boldsymbol{\lambda}) = \mathcal{B}_{\mathcal{D}}(\boldsymbol{\lambda})\mathcal{F}_{\mathcal{D}}(\boldsymbol{\lambda}) \\ &= -4\left(c_2(\eta)\frac{d^2}{d\eta^2} + \left(c_1(\eta,\boldsymbol{\lambda}^{[M,N]}) - 2c_2(\eta)\frac{\partial_{\eta}\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}\right)\frac{d}{d\eta}\right. \\ &\quad \left.+ c_2(\eta)\frac{\partial_{\eta}^2\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})} - c_1(\eta,\boldsymbol{\lambda}^{[M,N]} - \boldsymbol{\delta})\frac{\partial_{\eta}\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}{\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})}\right), \end{aligned} \quad (5.34)$$

$$\tilde{\mathcal{H}}_{\mathcal{D}}(\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) = \mathcal{E}(n;\boldsymbol{\lambda})P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}). \quad (5.35)$$

Since all the zeros of $\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})$ are simple for generic couplings, (5.35) is a Fuchsian differential equation for the J case. The characteristic exponents at the zeros of $\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})$ are the same everywhere, 0 and 3. The multi-indexed polynomials $\{P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda})\}$ provide infinitely many global solutions of the above Fuchsian equation (5.35) with $3+\ell$ regular singularities for the J case [47]. The L case is obtained as a confluent limit. These situations are basically the same as those of the exceptional polynomials.

For special choices of \mathcal{D} and by fine tuning the couplings, it is possible to construct a denominator polynomial $\Xi_{\mathcal{D}}(\eta;\boldsymbol{\lambda})$ having higher zeros. Explicit examples of double zeros were constructed in [62, 66, 69]. These can be considered as the *confluence of apparent singularities*.

Although we have restricted $d_j^{\text{I,II}} \geq 1$, there is no obstruction for deletion of $d_j^{\text{I,II}} = 0$. In terms of the multi-indexed polynomial (5.16), the level 0 deletions imply the following:

$$\begin{aligned} P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) \Big|_{d_M^{\text{I}}=0} &= P_{\mathcal{D}',n}(\eta;\boldsymbol{\lambda}-\tilde{\boldsymbol{\delta}}^{\text{I}}) \times A, \\ \mathcal{D}' &= \{d_1^{\text{I}}-1, \dots, d_{M-1}^{\text{I}}-1, d_1^{\text{II}}+1, \dots, d_N^{\text{II}}+1\}, \\ P_{\mathcal{D},n}(\eta;\boldsymbol{\lambda}) \Big|_{d_N^{\text{II}}=0} &= P_{\mathcal{D}',n}(\eta;\boldsymbol{\lambda}-\tilde{\boldsymbol{\delta}}^{\text{II}}) \times B, \end{aligned} \quad (5.36)$$

$$\mathcal{D}' = \{d_1^{\text{I}} + 1, \dots, d_M^{\text{I}} + 1, d_1^{\text{II}} - 1, \dots, d_{N-1}^{\text{II}} - 1\}, \quad (5.37)$$

where the multiplicative factors A and B are

$$A = \begin{cases} (-1)^M \prod_{j=1}^N (d_j^{\text{II}} + 1) & : \text{L} \\ -(-2)^{-M} \prod_{j=1}^{M-1} (g - h + d_j^{\text{I}} + 1) \cdot (-2)^{-N} \prod_{j=1}^N (d_j^{\text{II}} + 1) \cdot (n + h - \frac{1}{2}) & : \text{J} \end{cases}, \quad (5.38)$$

$$B = \begin{cases} (-1)^M \prod_{j=1}^M (d_j^{\text{I}} + 1) \cdot (n + g - \frac{1}{2}) & : \text{L} \\ 2^{-M} \prod_{j=1}^M (d_j^{\text{I}} + 1) \cdot (-2)^{-N} \prod_{j=1}^{N-1} (h - g + d_j^{\text{II}} + 1) \cdot (n + g - \frac{1}{2}) & : \text{J} \end{cases}. \quad (5.39)$$

From (5.22), $\Xi_{\mathcal{D}}$ behaves similarly. Therefore the level 0 deletion corresponds to $M + N - 1$ virtual states deletions. This is why we have restricted $d_j^{\text{I,II}} \geq 1$.

These relations (5.36)–(5.37) can be used for studying the equivalence of $\mathcal{H}_{\mathcal{D}}$. See [77, 78] for recent interesting developments on various equivalences.

The exceptional X_{ℓ} orthogonal polynomials of type I and II, [38, 39, 40, 42, 47, 49] correspond to the simplest cases of one virtual state deletion of that type, $\mathcal{D} = \{\ell^{\text{I}}\}$ or $\{\ell^{\text{II}}\}$, $\ell \geq 1$:

$$\xi_{\ell}(\eta; \boldsymbol{\lambda}) = \Xi_{\mathcal{D}}(\eta; \boldsymbol{\lambda} + \ell \boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}), \quad P_{\ell,n}(\eta; \boldsymbol{\lambda}) = P_{\mathcal{D},n}(\eta; \boldsymbol{\lambda} + \ell \boldsymbol{\delta} + \tilde{\boldsymbol{\delta}}) \times A, \quad (5.40)$$

where $\tilde{\boldsymbol{\delta}} = \tilde{\boldsymbol{\delta}}^{\text{I,II}}$ and the multiplicative factor A is $A = -1$ for $X^{\text{I}}L$, $(n + g + \frac{1}{2})^{-1}$ for $X^{\text{II}}L$, $2(n + h + \frac{1}{2})^{-1}$ for $X^{\text{I}}J$ and $-2(n + g + \frac{1}{2})^{-1}$ for $X^{\text{II}}J$. Most formulas between (5.23) and (5.35) look almost the same as those appearing in the theory of the exceptional orthogonal polynomials [40, 42, 47, 49, 50].

5.3 Duality between pseudo virtual states and eigenstates

For known shape-invariant potentials, Darboux transformations in terms of multiple *pseudo virtual state wavefunctions* are equivalent to Krein-Adler transformations deleting multiple eigenstates with *shifted parameters*. See Fig.4 for the illustration. Let us introduce appropriate symbols and notation for stating the duality or equivalence. As before, let $\mathcal{D} \stackrel{\text{def}}{=} \{d_1, d_2, \dots, d_M\}$ ($d_j \in \mathbb{Z}_{\geq 0}$) be a set of distinct non-negative integers. We introduce an integer N and fix it to be not less than the maximum of \mathcal{D} :

$$N \geq \max(\mathcal{D}). \quad (5.41)$$

Let us define another set of distinct non-negative integers $\bar{\mathcal{D}} = \{0, 1, \dots, N\} \setminus \{\bar{d}_1, \bar{d}_2, \dots, \bar{d}_M\}$ together with the shifted parameters $\bar{\boldsymbol{\lambda}}$:

$$\bar{\mathcal{D}} \stackrel{\text{def}}{=} \{0, 1, \dots, \check{\bar{d}}_1, \dots, \check{\bar{d}}_2, \dots, \check{\bar{d}}_M, \dots, N\} = \{e_1, e_2, \dots, e_{N+1-M}\},$$

$$\bar{d}_j \stackrel{\text{def}}{=} N - d_j, \quad \bar{\lambda} \stackrel{\text{def}}{=} \lambda - (N + 1)\delta. \quad (5.42)$$

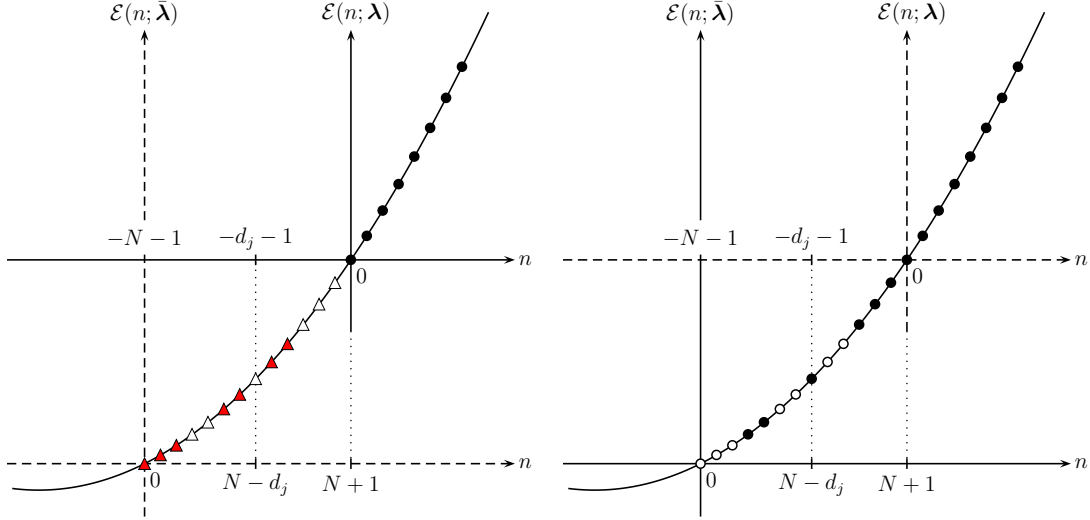


Figure 4: The left represents the Darboux transformations in terms of pseudo virtual states. The right corresponds to the Krein-Adler transformations in terms of eigenstates with shifted parameters. The black circles denote eigenstates. The white circles in the right graphic denote deleted eigenstates. The white triangles in the left graphic denote the pseudo virtual states used in the Darboux transformations. The red triangles denote the unused pseudo virtual states.

Starting from a shape-invariant original system (2.1) with the parameters λ , the system after Darboux transformations in terms of a set of pseudo virtual state wavefunctions \mathcal{D} is described by the Hamiltonian \mathcal{H}^{DP}

$$\begin{aligned} \mathcal{H}^{\text{DP}} &= -\frac{d^2}{dx^2} + U^{\text{DP}}(x), \\ U^{\text{DP}}(x) &= U(x; \lambda) - 2\partial_x^2 \log |W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \tilde{\phi}_{d_M}](x; \lambda)|. \end{aligned} \quad (5.43)$$

General theory presented in § 2.5 states that, if the Hamiltonian \mathcal{H}^{DP} is non-singular, the eigenstates are given by Φ_n^{DP} and $\check{\Phi}_j^{\text{DP}}$:

$$\begin{aligned} \Phi_n^{\text{DP}}(x) &= \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \tilde{\phi}_{d_M}, \phi_n](x; \lambda)}{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \tilde{\phi}_{d_M}](x; \lambda)} \quad (n = 0, 1, \dots), \\ \check{\Phi}_j^{\text{DP}}(x) &= \frac{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \check{\phi}_{d_j}, \dots, \tilde{\phi}_{d_M}](x; \lambda)}{W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \tilde{\phi}_{d_M}](x; \lambda)} \quad (j = 1, 2, \dots, M), \\ \mathcal{H}^{\text{DP}} \Phi_n^{\text{DP}}(x) &= \mathcal{E}(n; \lambda) \Phi_n^{\text{DP}}(x), \quad \mathcal{H}^{\text{DP}} \check{\Phi}_j^{\text{DP}}(x) = \mathcal{E}(-d_j - 1; \lambda) \check{\Phi}_j^{\text{DP}}(x). \end{aligned} \quad (5.44)$$

The system after Krein-Adler transformations in terms of $\bar{\mathcal{D}}$ with shifted parameters $\bar{\lambda}$ is described by the Hamiltonian \mathcal{H}^{KA}

$$\mathcal{H}^{\text{KA}} = -\frac{d^2}{dx^2} + U^{\text{KA}}(x),$$

$$U^{\text{KA}}(x) = U(x; \bar{\lambda}) - 2\partial_x^2 \log |W[\phi_0, \phi_1, \dots, \check{\phi}_{\bar{d}_1}, \dots, \check{\phi}_{\bar{d}_M}, \dots, \phi_N](x; \bar{\lambda})|. \quad (5.45)$$

Here we assume that the original system (2.1) with the shifted parameters $\bar{\lambda}$ has square integrable eigenstates, etc. If the Krein-Adler conditions are fulfilled, eigenstates are given by Φ_n^{KA} and $\check{\Phi}_j^{\text{KA}}$:

$$\begin{aligned} \Phi_n^{\text{KA}}(x) &= \frac{W[\phi_0, \phi_1, \dots, \check{\phi}_{\bar{d}_1}, \dots, \check{\phi}_{\bar{d}_M}, \dots, \phi_N, \phi_{N+1+n}](x; \bar{\lambda})}{W[\phi_0, \phi_1, \dots, \check{\phi}_{\bar{d}_1}, \dots, \check{\phi}_{\bar{d}_M}, \dots, \phi_N](x; \bar{\lambda})} \quad (n = 0, 1, \dots,), \\ \check{\Phi}_j^{\text{KA}}(x) &= \frac{W[\phi_0, \phi_1, \dots, \check{\phi}_{\bar{d}_1}, \dots, \check{\phi}_{\bar{d}_j}, \dots, \check{\phi}_{\bar{d}_M}, \dots, \phi_N](x; \bar{\lambda})}{W[\phi_0, \phi_1, \dots, \check{\phi}_{\bar{d}_1}, \dots, \check{\phi}_{\bar{d}_M}, \dots, \phi_N](x; \bar{\lambda})} \quad (j = 1, 2, \dots, M), \\ \mathcal{H}^{\text{KA}} \Phi_n^{\text{KA}}(x) &= \mathcal{E}(N+1+n; \bar{\lambda}) \Phi_n^{\text{KA}}(x), \quad \mathcal{H}^{\text{KA}} \check{\Phi}_j^{\text{KA}}(x) = \mathcal{E}(\bar{d}_j; \bar{\lambda}) \check{\Phi}_j^{\text{KA}}(x). \end{aligned} \quad (5.46)$$

The duality or the equivalence is stated as the following

Theorem. *The two systems with \mathcal{H}^{DP} and \mathcal{H}^{KA} are equivalent. To be more specific, the equality of the potentials and the eigenfunctions read:*

$$U^{\text{DP}}(x) - \mathcal{E}(-N-1; \lambda) = U^{\text{KA}}(x), \quad (5.47)$$

$$\Phi_n^{\text{DP}}(x) \propto \Phi_n^{\text{KA}}(x) \quad (n = 0, 1, \dots,), \quad (5.48)$$

$$\check{\Phi}_j^{\text{DP}}(x) \propto \check{\Phi}_j^{\text{KA}}(x) \quad (j = 1, 2, \dots, M). \quad (5.49)$$

The singularity free conditions of the potential are

$$\prod_{j=1}^{N+1-M} (m - e_j) \geq 0 \quad (\forall m \in \mathbb{Z}_{\geq 0}). \quad (5.50)$$

For $M = 1$, $\mathcal{D} = \{d_1\}$, $\bar{\mathcal{D}} = \{0, 1, \dots, \bar{d}_1, \dots, N\}$, the above conditions are satisfied by even d_1 , $d_1 \in 2\mathbb{Z}_{\geq 0}$. In other words, the pseudo virtual state wavefunctions $\{\tilde{\phi}_v\}$ for even v are nodeless. The above equalities (up to multiplicative factors) (5.47)–(5.49) ((5.48) with $n \in \mathbb{Z}_{\geq 0}$) are algebraic and they hold irrespective of the non-singularity conditions (5.50).

On top of the three fundamental potentials introduced in §2.6, the duality holds also for other shape invariant and thus exactly solvable potentials; Coulomb potential plus the centrifugal barrier, Kepler problem in spherical space, Morse potential, soliton potential, Rosen-Morse potential, Hyperbolic symmetric top II, Kepler problem in hyperbolic space, hyperbolic Pöschl-Teller potential [72].

We refer to [53, 72] for proofs and related discussions. When the eigenfunctions have the factorised form as shown in (2.61), which is true for H, L and J, the following polynomial Wronskian identities hold

$$W[\xi_{d_1}, \xi_{d_2}, \dots, \xi_{d_M}](\eta; \boldsymbol{\lambda}) \propto W[P_0, P_1, \dots, \check{P}_{\check{d}_1}, \dots, \check{P}_{\check{d}_M}, \dots, P_N](\eta; \bar{\boldsymbol{\lambda}}). \quad (5.51)$$

Verifying these identities for various choices of $\mathcal{D} = \{d_1, \dots, d_M\}$ for the three examples of pseudo virtual state wave functions (5.9)–(5.11) in §5.1.2 is left to readers as an exercise.

6 Exactly Solvable Scattering Problems

6.1 Scattering Problems

For non-confining potentials, when the physical regions extend to infinity, the systems have continuous spectrum as well as discrete eigenstates. We adopt the convention that the potentials vanish at infinity,² so that the plane waves $e^{\pm ikx}$, $k \in \mathbf{R}_+$, are the solutions of the Schrödinger equation with the positive energy $\mathcal{E} = k^2$ in the asymptotic regions. There are two types, the full line $x_1 = -\infty$, $x_2 = +\infty$ case called Group (A) or a half line $x_1 = 0$, $x_2 = +\infty$ case called Group (B). For the continuous spectrum states, we consider the scattering problems. Fig.5 shows the general situation of the full line scattering problem. On top of determining the discrete eigenstates (2.2) as above, we need to determine the transmission amplitude $t(k)$ (Group (A)) and the reflection amplitude $r(k)$ (both Group (A) and (B)) through the asymptotic behaviours of the wave function $\psi_k(x)$:

$$\mathcal{H}\psi_k(x) = k^2\psi_k(x), \quad k \in \mathbf{R}_+, \quad (6.1)$$

$$\psi_k(x) \approx \begin{cases} e^{ikx} & x \rightarrow +\infty \\ A(k)e^{ikx} + B(k)e^{-ikx} & x \rightarrow -\infty \end{cases} \quad (\text{A}), \quad (6.2)$$

$$\psi_k(x) \approx \begin{cases} r(k)e^{ikx} + e^{-ikx} & x \rightarrow +\infty, \end{cases} \quad (\text{B}). \quad (6.3)$$

For the full line scattering (Group (A)) one sends a unit amplitude of the right moving wave (e^{ikx}) at $x = -\infty$. Then a reflected left moving wave e^{-ikx} with the amplitude $r(k) \stackrel{\text{def}}{=} B(k)/A(k)$ is observed at $x = -\infty$ and a transmitted right moving wave (e^{ikx}) with the amplitude $t(k) \stackrel{\text{def}}{=} 1/A(k)$ is observed at $x = +\infty$. For the half line scattering (Group (B)), there is no transmitted wave.

²The cases where $U(-\infty) \neq U(+\infty)$ must be treated separately. See for example Rosen-Morse potential [3, 4, 71, 72, 76].

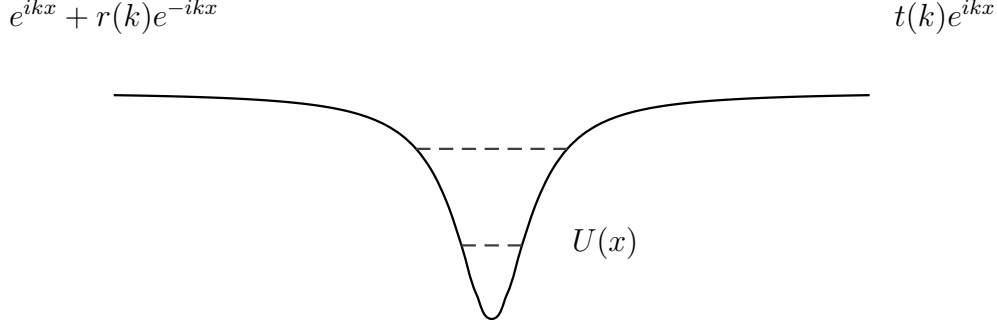


Figure 5: The general image of a full line scattering problem with the potential $U(x)$. The right going wave with the unit amplitude e^{ikx} is injected at $-\infty$. The reflected wave e^{-ikx} at $-\infty$ has an amplitude $r(k)$ and the transmitted wave e^{ikx} at $+\infty$ has an amplitude $t(k)$. The dashed lines show the discrete eigenlevels.

Since the wave function $\psi_k(x)$ (6.1) is an analytic function of the wave number k , the reflection $r(k)$ and the transmission $t(k)$ amplitudes are *meromorphic functions* of k . The above asymptotic behaviours should be compared with those of the discrete eigenstates:

$$\mathcal{H}\phi_n(x) = \mathcal{E}(n)\phi_n(x), \quad \mathcal{E}(0) < \mathcal{E}(1) < \mathcal{E}(2) < \dots < \mathcal{E}(n_{\max}) < 0, \quad (6.4)$$

$$\phi_n(x) \approx c_{\pm} e^{\mp \sqrt{-\mathcal{E}(n)}x}, \quad x \rightarrow \pm\infty. \quad (6.5)$$

By comparing the asymptotic behaviours (6.2), (6.3), (6.5), one finds that the *zeros* of $A(k)$, *i.e.* the *poles* of the transmission $t(k)$ and the reflection amplitude $r(k)$ on the *positive imaginary axis* $k = i\kappa$, $\kappa \in \mathbf{R}_+$ correspond to the discrete spectrum:

$$r(k) \approx \frac{\text{const}}{k - i\kappa}, \quad \kappa \in \mathbf{R}_+, \quad \exists n \in \{0, \dots, n_{\max}\}, \quad -\kappa^2 = \mathcal{E}(n). \quad (6.6)$$

The scattering amplitudes of shape invariant systems satisfy the constraints of shape invariance [76, 84]:

$$\text{full line : } t(k; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \left(\frac{ik + W_+}{ik + W_-} \right) t(k; \boldsymbol{\lambda}), \quad r(k; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \left(\frac{-ik + W_-}{ik + W_-} \right) r(k; \boldsymbol{\lambda}), \quad (6.7)$$

$$\text{half line : } r(k; \boldsymbol{\lambda} + \boldsymbol{\delta}) = \left(\frac{ik + W_+}{-ik + W_+} \right) r(k; \boldsymbol{\lambda}), \quad (6.8)$$

$$W_+ \stackrel{\text{def}}{=} - \lim_{x \rightarrow +\infty} \frac{\partial_x \phi_0(x; \boldsymbol{\lambda})}{\phi_0(x; \boldsymbol{\lambda})}, \quad W_- \stackrel{\text{def}}{=} - \lim_{x \rightarrow -\infty} \frac{\partial_x \phi_0(x; \boldsymbol{\lambda})}{\phi_0(x; \boldsymbol{\lambda})}. \quad (6.9)$$

These are simply obtained by evaluating the shape invariance relation

$$\psi_k(x; \boldsymbol{\lambda} + \boldsymbol{\delta}) \propto \psi_k^{(1)}(x; \boldsymbol{\lambda}) = \left(\frac{d}{dx} - \frac{\partial_x \phi_0(x; \boldsymbol{\lambda})}{\phi_0(x; \boldsymbol{\lambda})} \right) \psi_k(x; \boldsymbol{\lambda})$$

asymptotically.

6.2 Reflectionless Potentials

Reflectionless potentials of Schrödinger equations [81] played a very important role in theoretical physics. With special time dependence, they describe soliton solutions [82] of Korteweg de Vries (KdV) equation. As can be easily understood reflectionless potentials must be everywhere negative. Let us consider a reflectionless potential $U_N(x)$, which has N discrete eigenstates:

$$\mathcal{H} = -\frac{d^2}{dx^2} + U_N(x), \quad \mathcal{H}\psi_k(x) = k^2\psi_k(x), \quad (6.10)$$

$$\mathcal{H}\phi_{N,j}(x) = \mathcal{E}_j\phi_{N,j}(x), \quad \mathcal{E}_j = -k_j^2, \quad j = 1, \dots, N, \quad 0 < k_1 < k_2 < \dots < k_N. \quad (6.11)$$

According to Kay and Moses [81], it has an expression

$$U_N(x) \stackrel{\text{def}}{=} -2\partial_x^2 \log u_N(x), \quad (6.12)$$

$$u_N(x) \stackrel{\text{def}}{=} \det A_N(x), \quad (A_N(x))_{mn} \stackrel{\text{def}}{=} \delta_{mn} + \frac{c_m e^{-(k_m+k_n)x}}{k_m + k_n}, \quad m, n = 1, \dots, N, \quad (6.13)$$

in which $\{c_m\}$ are arbitrary positive parameters. A special choice of t -dependence of $\{c_m\}$

$$c_j \rightarrow c_j e^{8k_j^3 t}, \quad j = 1, \dots, N,$$

changes the reflectionless potential $U_N(x)$ to an N -soliton solution $U_N(x; t)$ of the KdV equation [82]:

$$\begin{aligned} U_N(x; t) &\stackrel{\text{def}}{=} -2\partial_x^2 \log u_N(x; t), \\ u_N(x; t) &\stackrel{\text{def}}{=} \det A_N(x; t), \quad (A_N(x; t))_{mn} \stackrel{\text{def}}{=} \delta_{mn} + \frac{c_m e^{-(k_m+k_n)x+8k_m^3 t}}{k_m + k_n}, \\ 0 &= \partial_t U_N - 6U_N \partial_x U_N + \partial_x^3 U_N. \end{aligned}$$

The very form of $U_N(x)$ (6.12) suggests that *the reflectionless potential can be obtained from the trivial potential $U(x) \equiv 0$ by multiple Darboux transformations* [80]. The Schrödinger equation with $U \equiv 0$ has *square non-integrable solutions*

$$\psi_j(x) \stackrel{\text{def}}{=} e^{k_j x} + \tilde{c}_j e^{-k_j x}, \quad 0 < k_1 < k_2 < \dots < k_N, \quad (-1)^{j-1} \tilde{c}_j > 0, \quad (6.14)$$

$$-\partial_x^2 \psi_j(x) = -k_j^2 \psi_j(x), \quad j = 1, \dots, N. \quad (6.15)$$

The inverses $\{1/\psi_j(x)\}$ are locally square integrable at $x = \pm\infty$. With the above sign of the parameters $\{\tilde{c}_j\}$ (6.14) the Wronskian of these seed solutions $\{\psi_j\}$ $W[\psi_1, \dots, \psi_N](x)$

is positive and it gives $u_N(x)$ up to a factor which is annihilated by ∂_x^2 after taking the logarithm:

$$W[\psi_1, \dots, \psi_N](x) = \prod_{j>l}^N (k_j - k_l) \cdot e^{\sum_{j=1}^N k_j x} u_N(x), \quad (6.16)$$

$$U_N(x) = -2\partial_x^2 \log W[\psi_1, \dots, \psi_N](x) = -2\partial_x^2 \log u_N(x). \quad (6.17)$$

Here we have redefined the coefficient of $e^{-2k_j x}$ in $u_N(x)$ to be $c_j/(2k_j)$, $c_j > 0$. Similar derivation of the reflectionless potential, without the eigenfunctions, was reported more than twenty years ago [83]. As explained in §2.5 (2.55) the above constructed $U_N(x)$ (6.17) has N -discrete eigenvalues $\mathcal{E}_j = -k_j^2$ with the eigenfunctions

$$\phi_{N,j}(x) \propto \frac{W[\psi_1, \dots, \check{\psi}_j, \dots, \psi_N](x)}{W[\psi_1, \dots, \psi_N](x)}, \quad j = 1, \dots, N. \quad (6.18)$$

By the same multiple Darboux transformation, the right moving plane wave solution e^{ikx} ($k > 0$) of the $U \equiv 0$ Schrödinger equation is mapped to

$$e^{ikx} \rightarrow \frac{W[\psi_1, \dots, \psi_N, e^{ikx}](x)}{W[\psi_1, \dots, \psi_N](x)} \sim \begin{cases} \prod_{j=1}^N (ik - k_j) \cdot e^{ikx} & x \rightarrow +\infty \\ \prod_{j=1}^N (ik + k_j) \cdot e^{ikx} & x \rightarrow -\infty \end{cases}, \quad (6.19)$$

as the Wronskian of exponential functions is a van der Monde determinant:

$$W[e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_M x}](x) = \prod_{1 \leq k < j \leq M} (\alpha_j - \alpha_k) \cdot e^{\sum_{j=1}^M \alpha_j x}. \quad (6.20)$$

This scattering wave solution has reflectionless asymptotic behaviour, which is consistent with (6.2) with $B(k) \equiv 0$. This is an alternative derivation of the reflection potential (6.12).

Its reflectionless property and the exact solvability are quite intuitively understood.

6.3 Extensions of Solvable Scattering Problems

Here we discuss extensions (deformations) of solvable scattering problems through multiple Darboux transformations in terms of polynomial type seed solutions $\{\tilde{\phi}_{d_j}(x)\}$, $j = 1, \dots, M$, indexed by a set of non-negative integers $\mathcal{D} = \{d_1, \dots, d_M\}$ which are the *degrees* of the polynomial parts of the seed solutions. The asymptotic behaviours of the polynomial type seed solution $\tilde{\phi}_v(x)$ are characterised by the *asymptotic exponents* Δ_v^\pm :

$$\tilde{\phi}_v(x) \approx \begin{cases} e^{x\Delta_v^+} & x \rightarrow +\infty \\ e^{x\Delta_v^-} & x \rightarrow -\infty \end{cases} \quad (\text{A}), \quad (6.21)$$

$$\tilde{\phi}_v(x) \approx e^{x\Delta_v^\dagger} \quad x \rightarrow +\infty, \quad (\text{B}). \quad (6.22)$$

The extensions by multiple Darboux transformations in terms of polynomial type seed solutions $\{\tilde{\phi}_{d_j}(x)\}$, $j = 1, \dots, M$ have been given in **Theorem** in §2.5, (2.53)–(2.55). One has to make sure that the *deformed potential is non-singular*. That requires the condition that the Wronskian $W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \tilde{\phi}_{d_M}](x)$ should not have any zeros in the interval $-\infty < x < \infty$ (A), or $0 < x < \infty$ (B). For the discrete eigenstates $\{\phi_{\mathcal{D},n}^{(M)}(x)\}$ and the scattering states $\{\psi_{\mathcal{D},k}^{(M)}(x)\}$ the transformation is iso-spectral. We stress, however, that additional discrete eigenstates may be created below the original ground state level $\mathcal{E}(0)$. Their number is equal to that of the used *pseudo virtual state wavefunctions*.

The deformation potential $-2\partial_x^2 \log|W[\tilde{\phi}_{d_1}, \tilde{\phi}_{d_2}, \dots, \tilde{\phi}_{d_M}](x)|$ vanishes asymptotically, $x \rightarrow \pm\infty$ for the seed solutions (6.21)–(6.22). The deformed continuous spectrum also starts at $\mathcal{E} = 0$ and the relationship between the energy \mathcal{E} and the wave number k , $\mathcal{E} = k^2$ is unchanged. The multi-indexed scattering amplitudes are easily obtained from the asymptotic form of the wavefunction $\psi_{\mathcal{D},k}^{(M)}(x)$ (2.55) by using the asymptotic forms of the original wavefunction $\psi_k(x)$ (6.2)–(6.3) and those of the polynomial seed solutions $\tilde{\phi}_v(x)$ (6.21)–(6.22). For the full line scattering (Group (A)) case, we obtain

$$\psi_{\mathcal{D},k}^{(M)}(x) \approx \prod_{j=1}^M (ik - \Delta_{d_j}^+) \cdot e^{ikx} \quad x \rightarrow +\infty, \quad (6.23)$$

$$\psi_{\mathcal{D},k}^{(M)}(x) \approx \prod_{j=1}^M (ik - \Delta_{d_j}^-) \cdot A(k) e^{ikx} + \prod_{j=1}^M (-ik - \Delta_{d_j}^-) \cdot B(k) e^{-ikx} \quad x \rightarrow -\infty, \quad (6.24)$$

which lead to multi-indexed multiplicative deformations of the transmission and reflection amplitudes:

$$(\text{A}) : \quad t_{\mathcal{D}}(k) = \prod_{j=1}^M \frac{k + i\Delta_{d_j}^+}{k + i\Delta_{d_j}^-} \cdot t(k), \quad r_{\mathcal{D}}(k) = (-1)^M \prod_{j=1}^M \frac{k - i\Delta_{d_j}^-}{k + i\Delta_{d_j}^-} \cdot r(k). \quad (6.25)$$

For the half line scattering (Group (B)) case, similar calculation gives

$$(\text{B}) : \quad \psi_{\mathcal{D},k}^{(M)}(x) \approx \prod_{j=1}^M (ik - \Delta_{d_j}^+) \cdot r(k) e^{ikx} + \prod_{j=1}^M (-ik - \Delta_{d_j}^+) \cdot e^{-ikx} \quad x \rightarrow +\infty, \quad (6.26)$$

$$(\text{B}) : \quad r_{\mathcal{D}}(k) = (-1)^M \prod_{j=1}^M \frac{k + i\Delta_{d_j}^+}{k - i\Delta_{d_j}^+} \cdot r(k). \quad (6.27)$$

The meromorphic character of the scattering amplitudes is preserved by the multi-indexed extensions. The added poles and zeros all appear on the imaginary k -axis determined solely by the asymptotic exponents of the used polynomial type seed solutions. The derivation depends on the simple fact: the Wronskian of exponential functions $W[e^{\alpha_1 x}, e^{\alpha_2 x}, \dots, e^{\alpha_M x}](x)$ is reduced to a van der Monde determinant (6.20), and most factors cancel out between the numerator and denominator of (2.54).

6.4 Example: Soliton Potential

Here we present one typical example of shape invariant and solvable potentials. That is, the soliton potential. For more examples, see [76]. The useful data are the eigenenergies, eigenfunctions, scattering data, *i.e.*, the transmission and reflection amplitudes, various polynomial seed solutions, the virtual and pseudo virtual state wavefunctions, the overshoot eigenfunctions together with the corresponding asymptotic exponents.

The soliton potential system has finitely many discrete eigenstates $0 \leq n \leq n_{\max}(\boldsymbol{\lambda}) = [h]'$ in the specified parameter range:

$$\boldsymbol{\lambda} = h, \quad \boldsymbol{\delta} = -1, \quad -\infty < x < \infty, \quad h > 1/2, \quad (6.28)$$

$$U(x; h) = -\frac{h(h+1)}{\cosh^2 x}, \quad \mathcal{E}(n; h) = -(h-n)^2, \quad \eta(x) = \tanh x, \quad (6.29)$$

$$\phi_n(x; h) = (\cosh x)^{-h+n} \times P_n^{(h-n, h-n)}(\tanh x), \quad W_+ = -W_- = h. \quad (6.30)$$

The transmission and reflection amplitudes are:

$$t(k; h) = \frac{\Gamma(-h-ik)\Gamma(1+h-ik)}{\Gamma(-ik)\Gamma(1-ik)}, \quad r(k; h) = \frac{\Gamma(ik)\Gamma(-h-ik)\Gamma(1+h-ik)}{\Gamma(-ik)\Gamma(-h)\Gamma(1+h)}. \quad (6.31)$$

The poles on the positive imaginary k -axis coming from the first Gamma function factor in the numerator of $t(k; h)$ (6.31), $-h-ik = -n$, $\Rightarrow k = i(h-n)$, $n = 0, 1, \dots, [h]'$ provide the eigenspectrum as above. As is well known, at *integer* $h = N \in \mathbb{Z}_{\geq 1}$ the *reflectionless potential* $r(k; h) \equiv 0$ is realised by the poles of the Gamma function $\Gamma(-N)$ in the denominator of $r(k; h)$. In fact, $U_N(x) = -N(N+1)/\cosh^2 x$ is a very special case of the generic reflectionless potential (6.12)–(6.13) for the choice of parameters

$$k_j = j, \quad c_j = \frac{(N+j)!}{j!(j-1)!(N-j)!}, \quad j = 1, \dots, N. \quad (6.32)$$

The potential and the scattering amplitudes (6.31) are invariant under the discrete transformation $h \rightarrow -(h+1)$, but the eigenvalues and the eigenfunctions are not. The relation

$|t(k; h)|^2 + |r(k; h)|^2 = 1$, $k \in \mathbf{R}_+$ holds. In this particular example, the scattering data $r(k; h)$ and $t(k; h)$ can be obtained by analytically continuing the eigenfunction $\phi_n(x; h)$ (6.30) through $ik \stackrel{\text{def}}{=} -h + n$, by rewriting the Jacobi polynomial $P_n^{(h-n, h-n)}(\tanh x)$ in terms of the Gauss hypergeometric function (2.63) and using its connection formulas.

Polynomial type seed solutions The discrete symmetry $h \rightarrow -h - 1$ generates the pseudo virtual wavefunctions, which lie below the ground state:

$$\begin{aligned} \text{pseudo virtual : } \tilde{\phi}_v(x; h) &= (\cosh x)^{h+1+v} P_v^{(-h-1-v, -h-1-v)}(\tanh x) \quad (v \in \mathbb{Z}_{\geq 0}), \\ \Delta_v^+ &= h + 1 + v > 0, \quad \Delta_v^- = -\Delta_v^+ < 0, \quad \tilde{\mathcal{E}}_v(h) = \mathcal{E}(-v - 1; h) < \mathcal{E}(0; h). \end{aligned} \quad (6.33)$$

The overshoot eigenfunctions [68] provide ‘pseudo’ virtual state wavefunctions for this potential for $v > 2h$:

$$\begin{aligned} \text{‘pseudo virtual’ : } \tilde{\phi}_v^{\text{os}}(x; h) &= \phi_v(x; h), \\ \Delta_v^+ &= -h + v > 0, \quad \Delta_v^- = -\Delta_v^+ < 0, \quad \tilde{\mathcal{E}}_v^{\text{os}}(h) < \mathcal{E}(0; h) \quad (v > 2h). \end{aligned} \quad (6.34)$$

Deformed scatterings A pseudo (‘pseudo’) virtual state wavefunction will add a new discrete eigenstate at its energy. It is trivial to verify that the pseudo (6.33) and ‘pseudo’ (6.34) virtual wavefunctions will add a pole on the positive imaginary k -axis at $k = i(v + h + 1)$, $k = i(v - h)$, respectively, with exactly the same energy of the employed seed solution, $-(h + v + 1)^2$ and $-(v - h)^2$, respectively. For both the pseudo (6.33) and ‘pseudo’ (6.34) virtual wavefunctions, $\Delta_v^+ = -\Delta_v^-$. This means that the deformation factors of the transmission and reflection amplitudes are the same except for a sign $(-1)^M$:

$$\frac{t_{\mathcal{D}}(k; h)}{t(k; h)} = (-1)^M \frac{r_{\mathcal{D}}(k; h)}{r(k; h)} = \prod_{j=1}^M \frac{k - i\Delta_{d_j}^-}{k + i\Delta_{d_j}^-}. \quad (6.35)$$

7 Summary and Comments

The basic structure and the recent developments in the theory of exactly solvable quantum mechanics are presented in an elementary way. Exactly solvable multi-particle dynamics, in particular, Calogero-Moser systems based on various root systems [9]–[11] could not be included. The concepts and methods of solvable quantum mechanics, the factorised Hamiltonians, Crum’s theorem and its modifications, generic Darboux transformations, shape

invariance, Heisenberg operator solutions, rational extensions in terms of polynomial type seed solutions, etc, can be generalised to *discrete quantum mechanics* [34]–[36], [85], in which Schrödinger equations are *difference equations*. We strongly believe that these new progress would be interesting to most readers.

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Appendix: Symbols, Definitions & Formulas

◦ shifted factorial (Pochhammer symbol) $(a)_n$:

$$(a)_n \stackrel{\text{def}}{=} \prod_{k=1}^n (a+k-1) = a(a+1) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (\text{A.1})$$

◦ hypergeometric series ${}_rF_s$:

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_r)_n}{(b_1, \dots, b_s)_n} \frac{z^n}{n!}, \quad (\text{A.2})$$

where $(a_1, \dots, a_r)_n \stackrel{\text{def}}{=} \prod_{j=1}^r (a_j)_n = (a_1)_n \cdots (a_r)_n$.

◦ differential equations

$$\text{H : } \partial_x^2 H_n(x) - 2x \partial_x H_n(x) + 2n H_n(x) = 0, \quad (\text{A.3})$$

$$\text{L : } x \partial_x^2 L_n^{(\alpha)}(x) + (\alpha + 1 - x) \partial_x L_n^{(\alpha)}(x) + n L_n^{(\alpha)}(x) = 0, \quad (\text{A.4})$$

$$\begin{aligned} \text{J : } (1-x^2) \partial_x^2 P_n^{(\alpha, \beta)}(x) + (\beta - \alpha - (\alpha + \beta + 2)x) \partial_x P_n^{(\alpha, \beta)}(x) \\ + n(n + \alpha + \beta + 1) P_n^{(\alpha, \beta)}(x) = 0. \end{aligned} \quad (\text{A.5})$$

◦ Rodrigues formulas

$$\text{H : } H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}, \quad (\text{A.6})$$

$$\text{L : } L_n^{(\alpha)}(x) = \frac{1}{n!} \frac{1}{e^{-x} x^\alpha} \left(\frac{d}{dx} \right)^n (e^{-x} x^{n+\alpha}). \quad (\text{A.7})$$

$$\text{J : } P_n^{(\alpha, \beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{1}{(1-x)^\alpha (1+x)^\beta} \left(\frac{d}{dx} \right)^n ((1-x)^{n+\alpha} (1+x)^{n+\beta}). \quad (\text{A.8})$$

References

- [1] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th edition, Oxford Univ. Press, Oxford (1963).
- [2] L. D. Landau and L. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd edition, Pergamon Press, Oxford (1965).
- [3] L. Infeld and T. E. Hull, “The factorization method,” *Rev. Mod. Phys.* **23** (1951) 21-68.
- [4] F. Cooper, A. Khare and U. Sukhatme, “Supersymmetry and quantum mechanics,” *Phys. Rep.* **251** (1995) 267-385.
- [5] A. Y. Morozov, A. M. Perelomov, A. A. Rosly, M. A. Shifman and A. V. Turbiner, “Quasiexactly solvable quantal problems: one-dimensional analog of rational conformal field theories,” *Int. J. Mod. Phys.* **A5** (1990) 803-832.
- [6] A. V. Turbiner, “Quasi-Exactly-Solvable Problems and $sl(2)$ Algebra,” *Comm. Math. Phys.* **118** (1988) 467-474.
- [7] A. G. Ushveridze, “Exact solutions of one- and multi-dimensional Schrödinger equations,” *Sov. Phys.-Lebedev Inst. Rep.* **2** (1988) 50, 54-58.
- [8] R. Sasaki and K. Takasaki, “Quantum Inozemtsev model, quasi-exact solvability and \mathcal{N} -fold supersymmetry,” *J. Phys.* **A34** (2001) 9533-9553, Corrigendum *J. Phys.* **A34** (2001) 10335, [arXiv:hep-th/0109008](#).
- [9] F. Calogero, “Solution of the one-dimensional N -body problem with quadratic and/or inversely quadratic pair potentials,” *J. Math. Phys.* **12** (1971) 419-436; B. Sutherland, “Exact results for a quantum many-body problem in one-dimension. II,” *Phys. Rev.* **A5** (1972) 1372-1376; J. Moser, “Three integrable Hamiltonian systems connected with isospectral deformations,” *Adv. Math.* **16** (1975) 197-220.
- [10] A. J. Bordner, N. S. Manton and R. Sasaki, “Calogero-Moser Models V: Supersymmetry, and Quantum Lax Pair,” *Prog. Theor. Phys.* **103** (2000) 463-487, [arXiv:hep-th/9910033](#).
- [11] S. P. Khastgir, A. J. Pocklington and R. Sasaki, “Quantum Calogero-Moser Models: Integrability for all Root Systems,” *J. Phys.* **A33** (2000) 9033-9064, [arXiv:hep-th/0005277](#).
- [12] G. Darboux, “Sur une proposition relative aux équations linéaires.” *C. R. Acad. Paris* **94** (1882) 1456-1459.

- [13] M. M. Crum, “Associated Sturm-Liouville systems,” *Quart. J. Math. Oxford Ser. (2)* **6** (1955) 121-127, [arXiv:physics/9908019](#).
- [14] E. Witten, “Dynamical breaking of supersymmetry,” *Nucl. Phys.* **B188** (1981) 513-554.
- [15] A. A. Andrianov, M. V. Ioffe and V. P. Spiridonov, “Higher derivative supersymmetry and the Witten index,” *Phys. Lett.* **A174** (1993) 273-279, [arXiv:hep-th/9303005](#).
- [16] S. M. Klishevich and M. S. Plyushchay, “Nonlinear supersymmetry on the plane in magnetic field and quasi-exactly solvable systems,” *Nucl. Phys.* **B616** (2001) 403-418, [arXiv:hep-th/0105135](#).
- [17] H. Aoyama, M. Sato and T. Tanaka, “General forms of an \mathcal{N} -fold supersymmetry family,” *Phys. Lett.* **B503** (2001) 423-429, [arXiv:quant-ph/0012065](#).
- [18] M. G. Krein, “On continuous analogue of Christoffel’s formula in orthogonal polynomial theory,” *Doklady Acad. Nauk. CCCP*, **113** (1957) 970-973.
- [19] V. É. Adler, “A modification of Crum’s method,” *Theor. Math. Phys.* **101** (1994) 1381-1386.
- [20] L. E. Gendenshtein, “Derivation of exact spectra of the Schroedinger equation by means of supersymmetry,” *JETP Lett.* **38** (1983) 356-359.
- [21] J. W. Dabrowska, A. Khare and U. P. Sukhatme, “Explicit wavefunctions for shape-invariant potentials by operator technique,” *J. Phys.* **A 21** (1988) L195-L200.
- [22] S. Odake and R. Sasaki, “Unified theory of annihilation-creation operators for solvable (‘discrete’) quantum mechanics,” *J. Math. Phys.* **47** (2006) 102102 (33pp), [arXiv:quant-ph/0605215](#).
- [23] S. Odake and R. Sasaki, “Exact solution in the Heisenberg picture and annihilation-creation operators,” *Phys. Lett.* **B641** (2006) 112-117, [arXiv:quant-ph/0605221](#).
- [24] G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge, (1999).
- [25] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Encyclopedia of mathematics and its applications, Cambridge Univ. Press, Cambridge, (2005).
- [26] T. S. Chihara, *An Introduction to orthogonal polynomials*, Gordon and Breach, New York (1978).
- [27] M. M. Nieto and L. M. Simmons, Jr., “Coherent States For General Potentials,” *Phys. Rev. Lett.* **41** (1978) 207-210; “Coherent States For General Potentials,” 1. Formalism, *Phys. Rev. D* **20** (1979) 1321-1331; 2. Confining One-Dimensional Examples, *Phys. Rev.*

- D **20** (1979) 1332-1341; 3. Nonconfining One-Dimensional Examples, Phys. Rev. D **20** (1979) 1342-1350.
- [28] S. Odake and R. Sasaki, “Exact Heisenberg operator solutions for multi-particle quantum mechanics,” J. Math. Phys. **48** (2007) 082106 (12 pp), [arXiv:0706.0768\[quant-ph\]](#).
 - [29] L. Vinet and A. Zhedanov, “Quasi-linear algebras and integrability (the Heisenberg picture),” SIGMA **4** (2008) 015 (22 pp), [arXiv:0802.0744\[math.QA\]](#).
 - [30] T. Fukui and N. Aizawa, “Shape-invariant potentials and an associated coherent state,” Phys. Lett. **A180** (1993) 308-313; J.-P. Gazeau and J.R. Klauder, “Coherent states for systems with discrete and continuous spectrum,” J. Phys. **A32** (1999) 123-132.
 - [31] E. Routh, “On some properties of certain solutions of a differential equation of the second order,” Proc. London Math. Soc. **16** (1884) 245-261.
 - [32] S. Bochner, “Über Sturm-Liouvillesche Polynomsysteme,” Math. Zeit. **29** (1929) 730-736.
 - [33] R. Koekoek and R. F. Swarttouw, “The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue,” [arXiv:math.CA/9602214](#); R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q -analogues*, Springer-Verlag (2010).
 - [34] S. Odake and R. Sasaki, “Discrete quantum mechanics,” (Topical Review) J. Phys. **A44** (2011) 353001 (47 pp), [arXiv:1104.0473\[math-ph\]](#).
 - [35] S. Odake and R. Sasaki, “Orthogonal Polynomials from Hermitian Matrices,” J. Math. Phys. **49** (2008) 053503 (43pp), [arXiv:0712.4106\[math.CA\]](#).
 - [36] S. Odake and R. Sasaki, “Exactly solvable ‘discrete’ quantum mechanics; shape invariance, Heisenberg solutions, annihilation-creation operators and coherent states,” Prog. Theor. Phys. **119** (2008) 663-700, [arXiv:0802.1075\[quant-ph\]](#).
 - [37] S. Yu. Dubov, V. M. Eleonskii and N. E. Kulagin, “Equidistant spectra of anharmonic oscillators,” Soviet Phys. JETP **75** (1992) 446-451; Chaos **4** (1994) 47-53.
 - [38] D. Gómez-Ullate, N. Kamran and R. Milson, “An extension of Bochner’s problem: exceptional invariant subspaces,” J. Approx Theory **162** (2010) 987-1006, [arXiv:0805.3376\[math-ph\]](#); “An extended class of orthogonal polynomials defined by a Sturm-Liouville problem,” J. Math. Anal. Appl. **359** (2009) 352-367. [arXiv:0807.3939\[math-ph\]](#).

- [39] C.Quesne, “Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry,” J. Phys. **A41** (2008) 392001 (6 pp), [arXiv:0807.4087\[quant-ph\]](#); B.Bagchi, C.Quesne and R.Roychoudhury, “Isospectrality of conventional and new extended potentials, second-order supersymmetry and role of PT symmetry,” Pramana J. Phys. **73** (2009) 337-347, [arXiv:0812.1488\[quant-ph\]](#).
- [40] S.Odake and R.Sasaki, “Infinitely many shape invariant potentials and new orthogonal polynomials,” Phys. Lett. **B679** (2009) 414-417, [arXiv:0906.0142\[math-ph\]](#).
- [41] C.Quesne, “Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics,” SIGMA **5** (2009) 084 (24 pp), [arXiv:0906.2331\[math-ph\]](#).
- [42] S.Odake and R.Sasaki, “Another set of infinitely many exceptional (X_ℓ) Laguerre polynomials,” Phys. Lett. **B684** (2010) 173-176, [arXiv:0911.3442\[math-ph\]](#). (Remark: J1(J2) in this reference corresponds to J2(J1) in later references.)
- [43] G.Junker and P.Roy, “Conditionally exactly solvable problems and nonlinear algebras,” Phys. Lett. **A232** (1997) 155-161; “Conditionally exactly solvable potentials: a supersymmetric construction method,” Ann. Phys. **270** (1998) 155-177.
- [44] B.Midya and B.Roy, “Exceptional orthogonal polynomials and exactly solvable potentials in position dependent mass Schroedinger Hamiltonians,” Phys. Lett. A **373** (2009) 4117-4122, [arXiv:0910.1209\[quant-ph\]](#).
- [45] D.Dutta and P.Roy, “Conditionally exactly solvable potentials and exceptional orthogonal polynomials,” J. Math. Phys. **51** (2010) 042101 (9 pp).
- [46] S.Odake and R.Sasaki, “Infinitely many shape invariant potentials and cubic identities of the Laguerre and Jacobi polynomials,” J. Math. Phys. **51** (2010) 053513 (9pp), [arXiv:0911.1585\[math-ph\]](#).
- [47] C.-L.Ho, S.Odake and R.Sasaki, “Properties of the exceptional (X_ℓ) Laguerre and Jacobi polynomials,” SIGMA **7** (2011) 107 (24 pp), [arXiv:0912.5447\[math-ph\]](#).
- [48] D.Gómez-Ullate, N.Kamran and R.Milson, “Exceptional orthogonal polynomials and the Darboux transformation,” J. Phys. **A43** (2010) 434016 (16 pp), [arXiv:1002.2666\[math-ph\]](#).
- [49] R.Sasaki, S.Tsujimoto and A.Zhedanov, “Exceptional Laguerre and Jacobi polynomials and the corresponding potentials through Darboux-Crum transformations,” J. Phys. **A43** (2010) 315204 (20pp), [arXiv:1004.4711\[math-ph\]](#).

- [50] S. Odake and R. Sasaki, “A new family of shape invariantly deformed Darboux-Pöschl-Teller potentials with continuous ℓ ,” J. Phys. A **44** (2011) 195203 (14pp), [arXiv:1007.3800\[math-ph\]](#).
- [51] C.-L. Ho, “Dirac(-Pauli), Fokker-Planck equations and exceptional Laguerre polynomials,” Ann. Phys. **326** (2011) 797-807, [arXiv:1008.0744\[math-ph\]](#).
- [52] D. Gómez-Ullate, N. Kamran and R. Milson, “On orthogonal polynomials spanning a non-standard flag,” Contem. Math. **563** (2011) 51-71, [arXiv:1101.5584\[math-ph\]](#).
- [53] L. García-Gutiérrez, S. Odake and R. Sasaki, “Modification of Crum’s Theorem for ‘Discrete’ Quantum Mechanics,” Prog. Theor. Phys. **124** (2010) 1-26, [arXiv:1004.0289\[math-ph\]](#).
- [54] C-L. Ho and R. Sasaki, “Zeros of the exceptional Laguerre and Jacobi polynomials,” ISRN Mathematical Physics **2012** 920475 (27pp), [arXiv:1102.5669\[math-ph\]](#).
- [55] D. Gómez-Ullate, N. Kamran and R. Milson, “Two-step Darboux transformations and exceptional Laguerre polynomials,” J. Math. Anal. Appl. **387** (2012) 410-418, [arXiv:1103.5724\[math-ph\]](#).
- [56] C.-L. Ho, “Prepotential approach to solvable rational potentials and exceptional orthogonal polynomials,” Prog. Theor. Phys. **126** (2011) 185-201, [arXiv:1104.3511\[math-ph\]](#).
- [57] S. Odake and R. Sasaki, “Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials,” Phys. Lett. **B702** (2011) 164-170, [arXiv:1105.0508\[math-ph\]](#).
- [58] C. Quesne, “Higher-order SUSY, exactly solvable potentials, and exceptional orthogonal polynomials,” J. Mod. Phys. A **26** (2011) 1843-1852, [arXiv:1106.1990\[math-ph\]](#).
- [59] K. Takemura, “Heun’s equation, generalized hypergeometric function and exceptional Jacobi polynomial,” [arXiv:arXiv:1106.1543\[math.CA\]](#)
- [60] A. Ramos, “On the new translational shape invariant potentials,” J. Phys. A. **44** (2011) 342001, [arXiv:1106.3732\[quant-ph\]](#).
- [61] C. Quesne, “Rationally-extended radial oscillators and Laguerre exceptional orthogonal polynomials in kth-order SUSYQM,” J. Mod. Phys. A **26** (2011) 5337-5347, [arXiv:1110.3958\[math-ph\]](#).
- [62] D. Gómez-Ullate, N. Kamran and R. Milson, “A conjecture on exceptional orthogonal polynomials,” Found. Comput. Math. **13** (2013) 615-656, [arXiv:1203.6857\[math-ph\]](#).

- [63] D. Gómez-Ullate, F. Marcellán and R. Milson, “Asymptotic behaviour of zeros of exceptional Jacobi and Laguerre polynomials,” *J. Math. Anal. Appl.* **399** (2013) 480 - 495, [arXiv:1204.2282 \[math.CA\]](#).
- [64] B. Midya, “Quasi-Hermitian Hamiltonians associated with exceptional orthogonal polynomials,” *Phys. Lett. A* **376** (2012) 2851-2854, [arXiv:1205.5860 \[math-ph\]](#).
- [65] S. Post, S. Tsujimoto and L. Vinet, “Families of superintegrable Hamiltonians constructed from exceptional polynomials,” [arXiv:1206.0480 \[math-ph\]](#).
- [66] R. Sasaki and K. Takemura, “Global solutions of certain second order differential equations with a high degree of apparent singularity,” *SIGMA* **8** (2012) 085 (18pp), [arXiv:1207.5302 \[math.CA\]](#).
- [67] C.-I. Chou and C.-L. Ho, “Generalized Rayleigh and Jacobi processes and exceptional orthogonal polynomials,” [arXiv:1207.6001 \[math-ph\]](#).
- [68] C. Quesne, “Revisiting (quasi-)exactly solvable rational extensions of the Morse potential,” *Int. J. Mod. Phys. A* **27** (2012) 1250073, [arXiv:1203.1812 \[math-ph\]](#); “Novel Enlarged Shape Invariance Property and Exactly Solvable Rational Extensions of the Rosen-Morse II and Eckart Potentials,” *SIGMA* **8** (2012) 080 (19pp), [arXiv:1208.6165 \[math-ph\]](#).
- [69] C.-L. Ho, R. Sasaki and K. Takemura, “Confluence of apparent singularities in multi-indexed orthogonal polynomials: the Jacobi case,” *J. Phys. A* **46** (2013) 115205 (21pp) [arXiv:1210.0207 \[math.CA\]](#).
- [70] I. Marquette and C. Quesne, “Two-step rational extensions of the harmonic oscillator: exceptional orthogonal polynomials and ladder operators,” *J. Phys. A* **46** (2013) 155201, [arXiv:1212.3474 \[math-ph\]](#).
- [71] S. Odake and R. Sasaki, “Extensions of solvable potentials with finitely many discrete eigenstates,” *J. Phys. A* **46** (2013) 235205 (15pp), [arXiv:1301.3980 \[math-ph\]](#).
- [72] S. Odake and R. Sasaki, “Krein-Adler transformations for shape-invariant potentials and pseudo virtual states,” *J. Phys. A* **46** (2013) 245201 (24pp), [arXiv:1212.6595 \[math-ph\]](#).
- [73] S. Odake, “Recurrence Relations of the Multi-Indexed Orthogonal Polynomials,” *J. Math. Phys.* **54** (2013) 083506, [arXiv:1303.5820 \[math-ph\]](#).
- [74] D. Gómez-Ullate, Y. Grandati and R. Milson, “Rational extensions of the quantum harmonic oscillator and exceptional Hermite polynomials,” *J. Phys. A* **47** (2014) 015203, [arXiv:1306.5143 \[math-ph\]](#).

- [75] D. Gómez-Ullate, Y. Grandati and R. Milson, “Extended Krein-Adler theorem for the translationally shape invariant potentials,” [arXiv:1309.3756\[nlin.SI\]](#).
- [76] C.-L. Ho, J.-C. Lee and R. Sasaki, “Scattering amplitudes for multi-indexed extensions of solvable potentials,” *Annals of Physics* **343** (2014) 115–131, [1309.5471\[quant-ph\]](#).
- [77] S. Odake, “Equivalences of the Multi-Indexed Orthogonal Polynomials,” *J. Math. Phys.* **55** (2014) 013502, [arXiv:1309.2346\[math-ph\]](#).
- [78] K. Takemura, “Multi-indexed Jacobi polynomials and Maya diagrams,” [arXiv:1311.3570\[math-ph\]](#).
- [79] Y. Grandati, “Exceptional orthogonal polynomials and generalized Schur polynomials,” [arXiv:1311.4530\[math-ph\]](#).
- [80] R. Sasaki, “Exactly solvable potentials with finitely many discrete eigenvalues of arbitrary choice,” *J. Math. Phys.* **55** (2014) 062101 (11pp), [arXiv:1402.5474\[math-ph\]](#).
- [81] I. Kay and H. M. Moses, “Reflectionless transmission through dielectrics and scattering potentials,” *J. Appl. Phys.* **27** (1956) 1503-1508.
- [82] R. Hirota, “Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons,” *Phys. Rev. Lett.* **27** (1971) 1192-1194.
- [83] V. B. Matveev and M. A. Salle, *Darboux transformations and solitons*, Springer-Verlag, Berlin Heidelberg (1991).
- [84] A. Khare and U. P. Sukhatme, “Scattering amplitudes for supersymmetric shape-invariant potentials by operator methods,” *J. Phys.* **A21** (1988) L501-L508.
- [85] S. Odake and R. Sasaki, “Unified theory of exactly and quasi-exactly solvable ‘discrete’ quantum mechanics: I. Formalism,” *J. Math. Phys.* **51** (2010) 083502 (24pp), [arXiv:0903.2604\[math-ph\]](#).